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Affine geometries defined by fiber preserving diffeomorphisms

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Abstract. *Clearly a given manifold M may support more than one metric tensor and generally one may select a particular metric on M via a variational procedure defined on the class of all metrics on M. Obviously the class of all metrics on M is a subset of the set of all sections of the vector bundle $T_2^0 M$ and thus one has a rigorous framework for any theory which has as its goal the selection of a metric in this way (in particular, general relativity is such a theory). It is our purpose to develop such a framework for affine geometry. We do not consider specific procedures to select an affine geometry analogous to the selection of a metric via the variation of some Lagrangian, but we establish the arena where such procedures would be meaningful. In the case of Riemannian geometry, this arena would be the set of all sections of the finite dimensional vector bundle $T_2^0 M$ and in this context it is important that covariant derivatives of such sections are again sections of the same bundle. Moreover, the covariant derivative of a given metric is relatively simple as it arises from a linear action of $G\ell(n, \mathbf{R})$ on a typical fiber of $T_2^0 M$. In the case of affine geometry we find that the appropriate arena is the set of all sections of an infinite dimensional vector bundle $\mathcal{V}(E, V, \delta)$. Moreover, since the group $Aff(\delta)$ relating «change of basis» is not compact and acts on the fiber of this infinite dimensional bundle, it turns out that the group action is not generally continuous when one uses the Whitney C^∞ topology on a typical fiber; rather one must use the Schwartz C^∞-topology widely used in the theory of distributions to obtain a differentiable action. Moreover, the action of the group on a typical fiber is not linear so that the usual formulas for covariant derivatives must be modified. An interesting consequence of our investigation is that the vector bundle $\mathcal{V}$ can be extended and the action of the group also extended so that the nonlinearity is only a «second order nonlinearity», i.e., formulas for the covariant derivative of a section involve only linear terms and bilinear terms (see Equation 4.2). In addition to this feature, formulas for covariant derivatives of affine geometries are developed which are fully analogous to those for covariant derivatives of Riemannian metrics*

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(see Theorem 4.2). Symmetry-breaking properties associated with special classes of affine structures are obtained and parallels are drawn with the metric case (recall that a metric reduces the linear frame bundle to the bundle of orthonormal frames, symmetry is broken from the general linear group to the orthogonal group). In the last section of the paper we show how our formalism relates to certain already-developed applications of affine geometry to charged particle dynamics worked out in more detail by Norris and his collaborators.

1. INTRODUCTION

It is the purpose of this paper to introduce a formalism which is adequate to undertake a development of affine geometry via techniques analogous to certain of those used in Riemannian geometry. Although affine geometry has received only a fraction of the attention bestowed upon its richer counterpart Riemannian geometry, it is the case that a corpus of material exists due largely to Cartan, the rudiments of which may be found in textbook form in [10] and [11]. The principal object of study in the present work is the class of all affine geometries defined on a given affine bundle. To our knowledge the concept of an affine bundle has only recently been cast into a rigorous form. The idea was rigorously treated by Crampin and Thompson in [2] who in turn attribute the formal definition to Goldschmidt [5]. There is no apparent overlap of our results with either of these papers. Even our definition differs slightly from theirs, but our version seems better adapted to our purpose of studying the class of all affine geometries on a given affine bundle.

It should be noted that the physics literature abounds with applications of affine geometry, especially to the so-called metric-affine theories of extended gravitation [7]. These theories were initiated for the most part in work of Cartan [1], but have been developed by a number of authors since that time. Some of this work utilizes various classes of affine geometries in an attempt to understand interactions between fields defined by various types of elementary particles and gravitational fields. The present paper derives much of its motivation from the affine geometrical unification of gravity and electromagnetism achieved in [9] and [13]. This being the case, certain of our ideas are formulated within a context which facilitates their application to these physical theories. In particular, practically all of our results could be formulated within the class of all affine bundles without reference to what we call pointed affine bundles, but we have chosen to carry along a global section of each of our affine bundles and keep track of how these sections relate to our various constructions, especially since such distinguished sections play a central role in the physical theories developed in [9] and [13].

On the other hand, we emphasize that our results are quite independent of these physical theories and we believe they have mathematical merit apart from any application. There are a number of ways in which the development of affine geometry differs from its Riemannian counterpart. The most significant departures seem related ultimately to certain nonlinear features of the theory. These nonlinear features force us away from the finite dimensional techniques which work so well when one studies Riemannian geometry and require the development of infinite dimensional techniques. Nevertheless, parallels between the two theories exist, even if only at a primitive level. It is our aim to expose this parallelism between the two theories. In particular, some of these somewhat primitive ideas of metric geometry which will find a counterpart in affine geometry are as follows:

(1) if g is an arbitrary metric on a manifold M then there is a vector bundle $T^{\circ}_{2}M$ such that g is a section of $T^{\circ}_{2}M$;

(2) the set of all sections g of the vector bundle $T^{\circ}_{2}M$ are in bijective correspondence with equivariant maps $\hat{g}$ from the frame bundle LM of M into the vector space $T^{\circ}_{2}\mathbf{R}^{m}(m = \dim M)$;

(3) the equivariant mapping $\hat{g} : LM \rightarrow T^{\circ}_{2}\mathbf{R}^{m}$ which arises from a metric g carries LM onto an orbit of a metric g_{o} on the vector space R^{m} and the type of the metric g is determined by the type of g_{o}, e.g., g is positive definite if g_{o} is and g is Lorentzian if g_{o} is;

(4) a metric g reduces the frame bundle LM to the subbundle of g-orthonormal frames of LM and, moreover, any connection ω on LM reduces to this subbundle iff $D^{\omega}\hat{g} = 0$;

(5) there are interesting formulas which involve the covariant derivative of a metric, in particular if X, Y, Z are vector fields on M then

$$(\nabla^{\omega}_{Z}g)(X,Y) = \nabla^{\omega}_{Z}(g(X,Y)) - g(\nabla^{\omega}_{Z}X,Y) - g(X,\nabla^{\omega}_{Z}Y) .$$

Although these properties admittedly capture only the grossest features of metric geometry, they do provide a framework within which different metric geometries may be studied. For example, in a theory in which metric geometry is itself a variable (such as general relativity) one requires an arena in which one can pose a procedure (such as the variational principle) for choosing a geometry with specified properties. Heretofore such an arena has not existed for affine geometries. The present paper addresses this problem by developing affine ideas which parallel the metric properties (1)–(5) above.

More specifically our principal object of concern is what we call an *affine bundle* (E, V, δ). Here E is a fiber bundle with base space a manifold M, V is a vector bundle also having M as base space, and δ is a function whose value at $p \in M$ is a difference function from $E_p \times E_p$ to V_p. We refer to δ as a *difference function field* and we show that δ plays a role for affine geometry analogous to the role g enjoys for metric geometry. We now briefly discuss the various affine counterparts to (1)–(5) above. Let (E, V, δ) be an affine bundle fixed once for all.

(1) For $p \in M$ the set $\mathcal{D}(E_p, V_p)$ of all difference functions from $E_p \times E_p$ to V_p is not a vector space with respect to pointwise operations. We find a vector space, which

we denote by $\mathcal{V}(E_p, V_p)$, that represents the «linear closure» of $\mathcal{D}(E_p, V_p)$. For technical reasons this vector space is further enlarged to a vector space $\mathcal{V}^2(E_p, V_p)$ which is closed under exterior differentiation. Natural topologies are found relative to which both $\mathcal{V}(E_p, V_p)$ and $\mathcal{V}^2(E_p, V_p)$ become Fréchet spaces. Both $\mathcal{V}(E, V) = \bigcup_{p\in M} \mathcal{V}(E_p, V_p)$ and $\mathcal{V}^2(E, V) := \bigcup_{p\in M} \mathcal{V}^2(E_p, V_p)$ are infinite dimensional vector bundles and difference function fields are sections of both bundles. It is clear that $\mathcal{V}(E, V)$ plays the role of $T_2^\circ M$ in this theory and that the nonlinear features force us into an infinite dimensional framework.

(2) For the given affine bundles (E, V, δ) one can build a finite-dimensional affine frame bundle AE having as structure group the group A_n of all affine transformations of a typical fiber of E. If $\tilde{E}$ and $\tilde{V}$ are standard fibers of E and V respectively then there is a bijection from the set of all sections of $\mathcal{V}^2(E, V)$ onto the set of all $\mathcal{V}^2(\tilde{E}, \tilde{V})$-valued equivariant maps defined on AE. Clearly the Fréchet space $\mathcal{V}^2(\tilde{E}, \tilde{V})$ plays a role analogous to $T_2^\circ \mathbf{R}^m$ in a metric theory.

(3) If α is a difference function field on E to V (which may or may not coincide with δ) and $\hat{\alpha}$ is its corresponding equivariant mapping from AE into $\mathcal{V}^2(\tilde{E}, \tilde{V})$, then $\hat{\alpha}$ may map AE onto an orbit of some $\delta_o \in \mathcal{D}(\tilde{E}, \tilde{V}) \subseteq \mathcal{V}^2(\tilde{E}, \tilde{V})$. If so, then δ_o in some sense defines the type of α but in this affine theory there are no canonical forms for δ_o in contrast to the metric case. This is due in part to the fact that δ_o is in general nonlinear. One may speculate whether some sort of invariants might be formulated in special cases classifying the various δ_o by their jets. We have not attempted such a classification but we have investigated those $\hat{\alpha}$ which map onto the orbit of an «affine difference function» from $\tilde{E} \times \tilde{E}$ to $\tilde{V}$. The set of «affine elements» of $\mathcal{D}(\tilde{E}, \tilde{V})$ is a finite-dimensional manifold and thus provides a tractable subset of $\mathcal{D}(\tilde{E}, \tilde{V})$ which may be utilized as a model for development in the general case.

(4) There is a formalism developed for general principal fiber bundles which gives a mechanism for reducing a bundle. This mechanism works for the affine frame bundle AE just as it does for frame bundles. One difference which occurs in the affine case is that true reduction may or may not occur depending on the isotropy subgroup of the element $\delta_o \in \mathcal{D}(\tilde{E}, \tilde{V})$ discussed in (3).

(5) We obtain a number of formulas, some of which parallel similar formulas for covariant derivatives of metrics. There is a larger variety of expressions for the covariant derivative of a difference function field than for a metric field simply because the bundles involved are infinite dimensional. Perhaps the simplest of these results states that if α is a difference function field from E to V and X is a vector field on M then

$$(\nabla_X \alpha)(\sigma, \mu) = \nabla_X(\alpha(\sigma, \mu)) - d\alpha(\nabla_X \sigma, \nabla_X \mu)$$

where σ and μ are sections of E.

Rather than describe our results in more detail, we prefer instead to indicate briefly one reason we find it appropriate to study such structures. In [13], L.K. Norris has shown how to obtain a geometrical unification of electromagnetism and gravity. In this paper Norris utilizes affine geometry instead of the various linear modifications of Riemannian geometry which form the basis of most attempts to obtain such a unification. A basic idea of Norris' work is that one should have a model for classical particles which allows for all possible configurations of the energy-momentum of a particle in addition to the usual configurations of position and velocity. If M is a manifold which represents all possible positions of a particle, then the vector space T_pM represents all possible velocity vectors at $p \in M$. Norris postulates an affine space $\hat{\Pi}_p$ whose elements are energy-momentum affine vectors and which represents the set of all energy-momentum configurations of a particle at p. He also shows that the affine properties of the geometry are an essential feature of the model theory. His ideas are extended further in [9] in which certain assumptions are required of the affine geometry but which are only partially understood. For example, the formula

$$D(\delta(\hat{\pi}, \hat{\sigma}) = \delta_o(D\hat{\pi}, D\hat{\sigma})$$

is utilized in the paper but it is not clear what restriction this places on the connection. One of the aims of the present paper is to provide a mathematical framework relative to which the basic assumptions in the physical theory developed in [9] and [13] may be given a mathematically rigorous formulation.

The paper consists of six sections. The first section develops the results we need about affine spaces. In it we also define the various function spaces which are needed in the rest of the paper as well as the relevant actions of the Lie group of affine transformations and its corresponding Lie algebra on these function spaces. In the second section we introduce the rigorous definition of an affine bundle along with the attendant infinite dimensional vector bundles of interest, we discuss the affine frame bundle of an affine bundle, and we establish that standard constructions of associated bundles are valid in this infinite dimensional setting. In this section we also establish the one-to-one correspondence between sections of $\mathcal{V}^2(E, V)$ and equivariant maps from AE into $\mathcal{V}^2(\tilde{E}, \tilde{V})$. In the third section we define the exterior covariant derivative of equivariant maps from AE to $\mathcal{V}^2(\tilde{E}, \tilde{V})$. We show how to use this to define the derivative of sections of $\mathcal{V}^2(E, V)$. We show how these formulas simplify in the special case when the difference function field being differentiated is «affinely related» to the difference function field which defines the structure on AE. We also discuss the symmetry-breaking properties of difference function fields in this setting. In the fifth section of the paper we consider an alternative way of formulating our results. Instead of considering difference function fields we consider functions $\varphi_p : E_p \to V_p$ which depend smoothly on $p \in M$. These maps and their covariant derivatives seem to encode both information found in the corresponding difference function field δ_φ as well as information obtained if one chooses an arbitrary section σ of E which we think of as a choice of origin in each affine space E_p, $p \in M$. Covariant derivatives of φ have terms which cancel out when one passes to the covariant derivative of the corresponding difference function field δ_p. This suggests that these objects may be more useful in physics since unwanted terms which appear in the new formalism could always be set to zero if they are not needed whereas they are not even present in the difference function formalism. In the sixth and final section of the paper we briefly relate our formalism to certain already-developed applications of affine geometry referred to above.

The author acknowledges his debt to L. K. Norris not only for the initial inspiration for the paper but also for enlightening conversations about affine geometry and its applications to physics. Thanks are also due to the participants of the Mathematical Physics seminar at NCSU for suggestions which led to improvements in the text.

Finally, the author wishes to express his appreciation to the referee of the paper. He not only provided a reference which eliminated a long uninteresting appendix to the paper, but he also provided comments which improved the paper as a whole and which led to a greatly improved version of Section 3.

2. FUNCTION SPACES ASSOCIATED WITH AFFINE GEOMETRY

In this section of the paper we develop some rather basic concepts at the affine space level which will be expanded in later sections by imposing them on each fiber of an appropriately defined fiber bundle.

Recall [3] that an ordered triple (A, V, δ) is called an *affine space* if δ is a function from $A \times A$ to V such that

(1) for $\eta, \xi, \zeta \in A$, $\quad \delta(\eta, \zeta) = \delta(\eta, \xi) + \delta(\xi, \zeta)$,
(2) for $\xi \in A$, the function $\delta_\xi : A \to V$ defined by $\delta_\xi(\eta) = \delta(\xi, \eta)$ is a bijection.

If (A, V, δ) is an affine space, then we say that δ is a *difference function on* A. The set of all difference functions on A with values in V is denoted by $\mathcal{D}(A, V)$. Notice that generally $\mathcal{D}(A, V)$ is not a vector space under pointwise definitions of addition and scalar multiplication. On the other hand, if we relax one of these conditions we do obtain a vector space. Let $\mathcal{V}(A, V)$ denote the set of all functions $\alpha : A \times A \to V$ such that $\alpha(\eta, \zeta) = \alpha(\eta, \xi) + \alpha(\xi, \zeta)$ for $\eta, \xi, \zeta \in A$. Define operations on $\mathcal{V}(A, V)$ by $(\alpha_1 + \alpha_2)(\eta, \zeta) = \alpha_1(\eta, \xi) + \alpha_2(\eta, \xi)$ and $(c\alpha)(\eta, \xi) = c\alpha(\eta, \xi)$ for $\alpha_1, \alpha_2, \alpha \in \mathcal{V}(A, V)$, $c \in \mathbf{R}$, $\eta, \xi \in A$. The set of functions $\mathcal{V}(A, V)$ is then a vector space and contains $\mathcal{D}(A, V)$. We will often find it convenient to work with functions from A to V rather than functions from $A \times A$ to V. Denote by $\mathcal{M}(A, V)$ the set of all functions from A to V and notice that $\mathcal{M}(A, V)$ is also a vector space under the pointwise operations defined by $(\phi_1 + \phi_2)(\xi) = \phi_1(\xi) + \phi_2(\xi)$, $(c\phi)(\xi) = c\phi(\xi)$ where

$\phi_1, \phi_2, \phi \in \mathcal{M}(A,V)$, $c \in \mathbb{R}$, $\xi \in A$. Consider the function $\alpha^* : \mathcal{M}(A,V) \to \mathcal{V}(A,V)$ defined by $\alpha^*(\phi)(\eta,\xi) = \phi(\xi) - \phi(\eta)$ for $\phi \in \mathcal{M}(A,V)$, $,\eta,\xi \in A$. It is obvious that $\alpha^*(\phi) \in \mathcal{V}(A,V)$ for each ϕ and that α^* is linear. Moreover, its kernel consists only of constant maps. Thus we have:

PROPOSITION 2.1. *The sequence*

$$0 \to V \xrightarrow{i} \mathcal{M}(A,V) \xrightarrow{\alpha^*} \mathcal{V}(A,V) \to 0$$

is a short exact sequence. Moreover, if $\mathcal{B}(A,V)$ *denotes the set of all bijections from* A *onto* V, *then* $\alpha^*(\mathcal{B}(A,V)) = \mathcal{D}(A,V)$. ∎

We leave the details to the reader but observe that one consequence of the proposition is the obvious but important fact that two bijections ϕ and ψ from A to V give rise to the same difference function $\delta = \alpha^*(\phi) = \alpha^*(\psi)$ iff their difference $\phi - \psi$ is constant.

In subsequent sections we will be required to consider manifold structures on A and V. If we assume V is finite dimensional, then each choice of a basis in V defines an isomorphism from V onto $\mathbb{R}^n$. The set of all such isomorphisms is an atlas for V and it is easy to see that all elements of V^* are smooth relative to the corresponding differentiable structure $\tilde{\mathcal{A}}$. Actually it is easy to show that if $\mathcal{A}$ is any atlas on V having the property that all elements of V^* are smooth relative to $\mathcal{A}$ then $\mathcal{A} \subseteq \tilde{\mathcal{A}}$. Thus there is one and only one differentiable structure on V with respect to which all elements of V^* are smooth. If we are given a specific difference function $\delta : A \times A \to V$ and a point $\xi \in A$, then there is one and only one differentiable structure on A for which $\delta_\xi : A \to V$ is a diffeomorphism. Moreover, if δ_ξ is a diffeomorphism for *some* $\xi \in A$, then clearly it is a diffeomorphism for *all* $\xi \in A$.

Assume then that we have a fixed difference function δ_o from $A \times A$ onto a finite dimensional vector space V and that this function fixes the manifold structure on A once and for all. Relative to this choice let $\mathcal{M}_s(A,V)$, $\mathcal{V}_s(A,V)$, $\mathcal{B}_s(A,V)$, and $\mathcal{D}_s(A,V)$ denote the set of all smooth functions in the sets $\mathcal{M}(A,V)$, $\mathcal{V}(A,V)$, $\mathcal{B}(A,V)$, and $\mathcal{D}(A,V)$ respectively. If α^*_s denotes the restriction of α^* to $\mathcal{M}_s$ (A,V), we have the obvious:

COROLLARY 2.2. *The sequence*

$$0 \to V \xrightarrow{i} \mathcal{M}_s(A,V) \xrightarrow{\alpha^*} \mathcal{V}_s(A,V) \to 0$$

is a short exact sequence and $\alpha^*_s(\mathcal{B}_s(A,V)) \subseteq \mathcal{D}_s(A,V)$. ∎

We have enlarged the space $\mathcal{D}_s(A,V)$ to $\mathcal{V}_s(A,V)$ because, in subsequent sections, we will find it convenient to know that certain group actions on $\mathcal{D}_s(A,V)$ actually arise as restrictions of a linear action on the vector space $\mathcal{V}_s(A,V)$. Thus we think of $\mathcal{V}_s(A,V)$ as a «linear closure» of $\mathcal{D}_s(A,V)$. It turns out that this «linear closure» of $\mathcal{D}_s(A,V)$ is also too restrictive. We will require also an enlarged space which is closed under differentiation, i.e., if $\alpha : A \times A \to V$ is in the space we also require that $d_{(\eta,\xi)}\alpha$ be in the space for $(\eta,\xi) \in A \times A$. Obviously this latter requirement may only be made subject to some identifications which we now clarify.

If $\alpha : A\times A \to V$ is a smooth function then $d_{(\eta,\xi)}\alpha$ is a function from $T_{(\eta,\xi)}(A\times A)$ into $T_{\alpha(\eta,\xi)}V$. Since V is a vector space we know that $T_{\alpha(\eta,\xi)}V$ may be identified with V, thus $d_{(\eta,\xi)}\alpha$ becomes a vector-valued 1-form on $A \times A$. On the other hand, our differentiable structure on A is itself inherited from the fixed difference function δ_o, and, for any fixed $\xi_o \in A$, $\delta_{\xi_o} := (\delta_o)_{\xi_o} : A \to V$ is a diffeomorphism. It follows that $(d_\eta \delta_{\xi_o})^{-1} \circ \delta_{\xi_o}$ is a function from A to $T_\eta A$ and thus

$$\big((d_\eta \delta_{\xi_o})^{-1} \circ \delta_{\xi_o})\big) \times \big((d_\xi \delta_{\xi_o})^{-1} \circ \delta_{\xi_o}\big)$$

is a function from $A\times A$ to $T_\eta A\times T_\xi A$. If we identify $d_{(\eta,\xi)}\alpha : T_\eta A\times T_\xi A \to V$ with the function $\tilde{d}_{(\eta,\xi)}\alpha := d_{(\eta,\xi)}\alpha \circ \big[\big((d_\eta \delta_{\xi_o})^{-1} \circ \delta_{\xi_o}\big) \times \big((d_\xi \delta_{\xi_o})^{-1} \circ \delta_{\xi_o}\big)\big]$ then we may regard $d_{(\eta,\xi)}\alpha$ as a map from $A\times A$ to V. Even with this identification $\alpha \in \mathcal{V}_s(A,V)$ will *not* guarantee that $d_{(\eta,\xi)}\alpha$ is in $\mathcal{V}_s(A,V)$; thus we enlarge our space so that this property holds in the enlarged space.

Let $\mathcal{V}^2(A,V)$ denote the set of all smooth functions $\alpha : A \times A \to V$ such that for some ordered pair $(\phi,\psi) \in \mathcal{M}_s(A,V) \times \mathcal{M}_s(A,V)$

$$\alpha(\eta,\xi) = \phi(\xi) - \psi(\eta)$$

for all $(\eta,\xi) \in A \times A$. Then each element of $\mathcal{V}_s(A,V)$ is determined by a single element of $\mathcal{M}_s(A,V)$ while each element of $\mathcal{V}^2_s(A,V)$ is determined by an ordered pair of elements of $\mathcal{M}_s(A,V)$. It is clear that $\mathcal{V}^2_s(A,V)$ is a vector space under the usual pointwise operations analogous to those used on $\mathcal{V}_s(A,V)$.

If $\pi_1, \pi_2 : A \times A \to A$ are the projections of $A \times A$ onto A, we see that $\alpha \in \mathcal{V}^2(A,V)$ implies that $\alpha = \phi\circ\pi_2 - \psi\circ\pi_1$ and consequently $d_{(\eta,\xi)}\alpha = d_\xi\phi\circ d_{(\eta,\xi)}\pi_2 - d_\eta\phi\circ d_{(\eta,\xi)}\pi_1$. But $\tilde{d}_{(\eta,\xi)}\alpha = (d_\xi\phi\circ(d_\xi\delta^{-1}_{\xi_o})\circ\delta_{\xi_o})\circ\pi_2 - (d_\eta\psi\circ(d_\eta\delta_{\xi_o})^{-1}\circ\delta_{\xi_o})\circ\pi_1$ which is clearly in $\mathcal{V}^2_s(A,V)$. Thus under the identification $\tilde{d}_{(\eta,\xi)}\alpha = d_{(\eta,\xi)}\alpha$ above we see that $\alpha \in \mathcal{V}^2_s(A,V)$ implies that $d_{(\eta,\xi)}\alpha \in \mathcal{V}^2_s(A,V)$. We have proven:

PROPOSITION 2.3. *The vector space of functions $\mathcal{V}^2_s(A,V)$ contains $\mathcal{V}_s(A,V)$ as a subspace and also has the property that $\alpha \in \mathcal{V}^2_s(A,V)$ implies that $d_{(\eta,\xi)}\alpha \in \mathcal{V}^2_s(A,V)$ for every pair $(\eta,\xi) \in A \times A$.*

Eventually we will define the covariant derivative of equivariant functions each of which has as domain a certain principal fiber bundle with structure group the group of affine transformations. The range of these maps is $\mathcal{V}_s^2(A,V)$. Since the covariant derivative is defined in terms of the action of the Lie algebra of the structure group on the range of the maps being differentiated, we must show how the Lie algebra of the group of affine transformations acts on $\mathcal{V}_s^2(A,V)$. This action is induced by an action of the group of affine transformations on $\mathcal{V}_s^2(A,V)$ which in turn is induced by actions of this same group on both A and V.

Assume that (A, V, δ_o) is a fixed affine space and recall that a mapping $f : A \to A$ is an affine mapping iff there exists a linear mapping $f_L : V \to V$ such that $\delta_o(f(\eta), f(\xi)) = f_L(\delta_o(\eta,\xi))$ for all $\eta, \xi \in A$ (see [3]). The set of all bijective affine mappings from A to A is denoted by $\mathrm{Aff}(\delta_o)$. This set of mappings is a group under composition of functions and there is a homomorphism from $\mathrm{Aff}(\delta_o)$ onto $G\ell(V)$ defined by $f \to f_L$. It is not hard to show that the kernel of this homomorphism is isomorphic to V under addition. Each choice of $\xi_o \in A$ yields a splitting of the sequence $V \longrightarrow \mathrm{Aff}(\delta_o) \longrightarrow G\ell(V)$ and the mapping $f \longrightarrow (\delta_{\xi_o}(f(\xi_o)), f_L)$ is an isomorphism from $\mathrm{Aff}(\delta_o)$ onto the semidirect product $V \rtimes G\ell(V)$ where the operation on $V \rtimes G\ell(V)$ is defined by

$$(v_1, g_1)(v_2, g_2) = (v_1 + g_1(v_2), g_1 \cdot g_2)$$

for $v_1, v_2 \in V$, $g_1, g_2 \in G\ell(V)$.

Observe that there are natural actions of $\mathrm{Aff}(\delta_o)$ on both A and on V. These actions are defined by $(f,\eta) \to f(\eta)$ and $(f, v) \to f_L(v)$, respectively (here $f \in \mathrm{Aff}(\delta_o)$, $\eta \in A$, $v \in V$). These actions clearly induce an action of $\mathrm{Aff}(\delta_o)$ on $\mathcal{V}_s^2(A,V)$ via $(f,\alpha) \to f\cdot\alpha$ where $f\cdot\alpha$ is defined by

$$(f\cdot\alpha)(\eta,\xi) = f_L(\alpha(f^{-1}(\eta), f^{-1}(\xi))) \tag{2.1}$$

for $f \in \mathrm{Aff}(\delta_o)$, $\alpha \in \mathcal{V}_s^2(A,V)$, $\eta,\xi \in A$.

Because of our interest in relating our results to the physical theories which in part motivate our work, we are interested in understanding the implications of a «choice of origin» in each affine space we consider. Such a choice of origin $\xi_o \in A$ provides us with the identification of $\mathrm{Aff}(\delta_o)$ with $V \rtimes G\ell(V)$ defined above. The actions of $\mathrm{Aff}(\delta_o)$ on the spaces A and V relative to this identification assume the form

$$(v,g)\cdot\eta = \delta_{\xi_o}^{-1}(g(\delta_{\xi_o}(\eta)) + v) \tag{2.2a}$$

$$(v,g)\cdot w = g(w) \tag{2.2b}$$

for $v \in V$, $w \in V$, $g \in G\ell(V)$, $\eta \in A$. When the action of $\mathrm{Aff}(\delta_o)$ on V is written in the form (2.2b) it is natural to ask whether it might be more appropriate to utilize another action of $\mathrm{Aff}(\delta_o)$ on V, namely the one defined by

$$(v,g)\cdot w = g(w) + v \tag{2.3}$$

where $(v,g) \in V \times G\ell(V)$ and $w \in V$.

It turns out that if we use (2.3) instead of (2.2) in definition (2.1), then $\mathcal{D}_s(A,V)$ will not be invariant; i.e., for $\alpha \in \mathcal{D}_s(A,V)$ and $a \in \mathrm{Aff}(\delta_o)$ it need not follow that $a \cdot \alpha \in \mathcal{D}_s(A,V)$. Generally, $\alpha \in \mathcal{D}_s(A,V)$ has the property that $\alpha(\eta,\eta) = 0$ for all $\eta \in A$ but $(a\cdot\alpha)(\eta,\eta)$ may not vanish. On the other hand, if we utilize (2.2) in the action defined by (2.1) we see that $\mathrm{Aff}(\delta_o)\cdot\mathcal{D}_s(A,V) \subseteq \mathcal{D}_s(A,V)$ as we require below.

Since we are interested in keeping track of the «choice of origin» in our affine spaces and in the subsequent implications of this choice, we find it useful to formalize the concept.

DEFINITION 2.1. To say that (A,V,δ_o,ξ_o) is a *pointed affine space* means that (A,V,δ_o) is an affine space and that $\xi_o \in A$. The point ξ_o will be called the *origin* of the affine space (A,V,δ_o). When $\mathrm{Aff}(\delta_o)$ acts on a pointed space (A,V,δ_o,ξ_o) we identify $\mathrm{Aff}(\delta_o)$ with $V \rtimes G\ell(V)$ as above and we identify the actions of $\mathrm{Aff}(\delta_o)$ on A, V, and $\mathcal{V}^2_s(A,V)$ with the actions defined by (2.1) and (2.2). Finally, when V is n-dimensional we denote $\mathrm{Aff}(\delta_o) \cong V \rtimes G\ell(V)$ by A_n and its Lie algebra by $a_n \cong V \rtimes g\ell(V)$. ■

Finally, to obtain the action of the Lie algebra on $\mathcal{V}^2_s(A,V)$ which we require below, one proceeds as usual to differentiate the corresponding actions of A_n on first A and then on $\mathcal{V}^2_s(A,V)$. If $t \to (v(t),g(t))$ is a one parameter group in A_n, then its corresponding element in a_n is $(\tilde v,\tilde g) = \frac{d}{dt}\big(v(t),g(t)\big)|_{t=0}$. Thus the action of a_n on A is defined by

$$\begin{aligned}(\tilde v,\tilde g)\cdot\xi &:= \frac{d}{dt}\big[\big(v(t),\, g(t)\big)\cdot\xi\big]|_{t=0}\\ &= \frac{d}{dt}\big[\delta_{\xi_o}^{-1}\big(g(t)\delta_{\xi_o}(\xi) + v(t)\big)\big]|_{t=0}\\ &= \big[d\delta_{\xi_o}^{-1}\circ\delta_{\xi_o}\big]\big(\delta_{\xi_o}^{-1}(\tilde g\delta_{\xi_o}(\xi) + \tilde v)\big)\end{aligned}$$

Recall that $d_\eta\delta_{\xi_o}^{-1}\circ\delta_{\xi_o}$ is the map we use to identify A with $T_\eta A$; thus we have that

$$(\tilde v,\tilde g)\cdot\xi := \delta_{\xi_o}^{-1}(\tilde g\delta_{\xi_o}(\xi) + \tilde v)\,. \tag{2.4}$$

We now proceed to derive a formula for the action of a_n on $\mathcal{V}_s^2(A,V)$. Let $\alpha \in \mathcal{V}_s^2(A,V)$ and let $\varphi, \psi \in \mathcal{M}_s(A,V)$ such that $\alpha = \varphi \circ \pi_2 - \psi \circ \pi_1$. If $t \to a(t) = (v(t), g(t))$ is a 1-parameter group in A_n and $\tilde{a} = \frac{da}{dt}(0)$ then

$$
\begin{aligned}
(\tilde{a} \cdot \alpha)(\eta,\xi) &:= \frac{d}{dt}\big([a(t) \cdot \alpha](\eta,\xi)\big)\big|_{t=0} \\
&= \frac{d}{dt}\big[g(t)\alpha\big(a(t)^{-1} \cdot \eta,\ a(t)^{-1} \cdot \xi\big)\big]\big|_{t=0} \\
&= \tilde{g} \cdot \varphi(\xi) - \tilde{g} \cdot \psi(\eta) + d_\xi\varphi(-\tilde{a} \cdot \xi) - d_\xi\psi(-\tilde{a} \cdot \eta) \\
&= \tilde{g} \cdot \alpha(\eta,\xi) - d_{(\eta,\xi)}\alpha(\tilde{a} \cdot \eta,\ \tilde{a} \cdot \xi)\ .
\end{aligned}
$$

Thus

$$
(2.5) \qquad (\tilde{a} \cdot \alpha)(\eta,\xi) = \tilde{g} \cdot \alpha(\eta,\xi) - d\alpha_{(\eta,\xi)}(\tilde{a} \cdot \eta,\ \tilde{a} \cdot \xi)\ .
$$

The formulas we have derived above are quite general and throughout the remainder of the paper we will be interested not only in this general case, but also in one special case which we have not yet discussed. This special case is concerned not with all maps from A to V but rather the affine maps from A to V.

DEFINITION 2.2. An element α of $\mathcal{V}_s^2(A,V)$ is said to be *affine* iff there exists affine maps $\varphi, \psi \in \mathcal{M}_s(A,V)$ such that $\alpha = \varphi \circ \pi_2 - \psi \circ \pi_1$. The set of all affine elements of $\mathcal{V}_s^2(A,V)$ will be denoted by $\mathrm{Aff}^2(A,V)$ and the set of all affine elements of $\mathcal{M}_s(A,V)$ will be denoted by $\mathrm{Aff}(A,V)$.

REMARK. Observe that if α is an affine element of $\mathcal{V}_s^2(A,V)$ and $\alpha \in \mathcal{D}_s(A,V)$, then it is appropriate to call α an *affine difference function* and in this case

$$
(2.6) \qquad \alpha = \varphi \circ \pi_2 - \varphi \circ \pi_1
$$

for some bijective affine map $\varphi : A \to V$.

PROPOSITION 2.4. *Assume that $\alpha \in \mathcal{V}_s^2(A,V)$ is affine and that $\alpha = \varphi \circ \pi_2 - \psi \circ \pi_1$ for $\varphi, \psi \in \mathrm{Aff}(A,V)$. If φ_L and ψ_L are the linear parts of φ and ψ, respectively, then*

(1) $\alpha = (\varphi_L \circ \delta_{\xi_o} \circ \pi_2) - (\psi_L \circ \delta_{\xi_o} \circ \pi_1) + v_\alpha$ *for some* $v_\alpha \in V$,

(2) $d\alpha = \tilde{d}\alpha = (\varphi_L \circ \tilde{\pi}_2) - (\varphi_L \circ \tilde{\pi}_2)$ *where $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are the maps from $A \times A$ to V defined by*

$$
\tilde{\pi}_1(\eta,\xi) = \delta_{\xi_o}(\eta), \quad \tilde{\pi}_2(\eta,\xi) = \delta_{\xi_o}(\xi)
$$

(3) for $\tilde{a} \in a_n$, $\tilde{a} = (\tilde{v}, \tilde{g})$, $\tilde{a} \cdot \alpha = [\tilde{g}, \varphi_L] \circ \tilde{\pi}_2 - [\tilde{g}, \psi_L] \circ \tilde{\pi}_1 - [\varphi_L(\tilde{v}) - \psi_L(\tilde{v})] + \tilde{g} v_\alpha$.

Proof. Statement (1) follows from Equation (2.6) and the definition of an affine mapping. Statement (2) follows from the canonical identification $\tilde{d}\alpha = d\alpha$ made in the paragraph immediately preceding the statement of Proposition 2.3 along with the fact that if $\ell : V \to V$ is linear then $d_v \ell = \ell$ for each $v \in V$. The proof of statement (3) is an easy consequence of equation (2.5) and statements (1) and (2). ∎

This completes our discussion of the various function spaces we will need at the purely affine space level. Our next section will develop these ideas further at the fiber bundle level.

3. AFFINE BUNDLES, CORRESPONDING PRINCIPAL BUNDLES AND VARIOUS ASSOCIATED BUNDLES

In this section we introduce the concept of an affine bundle. We study its bundle of affine frames and show that the affine bundle may be recovered as a bundle associated to the affine frame bundle and an appropriate action of A_n. This result is used to obtain a bijection from the set of all sections of an affine bundle (E, V, δ) onto the set of equivariant mappings from its bundle AE of affine frames into an appropriate Fréchet space.

DEFINITION 3.1. Let $(\tilde{E}, \tilde{V}, \tilde{\delta})$ be an affine space and M a manifold. Also let (E, V, δ) be an ordered triple such that

(1) (V, M, π_V) is a vector bundle over M with fiber $\tilde{V}$,

(2) (E, M, π_E) is a fiber bundle over M with fiber $\tilde{E}$, and

(3) for each $p \in M$, (E_p, V_p, δ_p) is an affine space.

One says that (E, V, δ) is a *trivial affine bundle with standard fiber* $(\tilde{E}, \tilde{V}, \tilde{\delta})$ iff both $(V, M, \pi_V) = (M \times \tilde{V}, M, \pi_V)$ and $(E, M, \pi_E) = (M \times \tilde{E}, M, \pi_E)$ are trivial fiber bundles and δ_p is given by $\delta_p((p, \tilde{\xi}), (p, \tilde{\eta})) = \tilde{\delta}(\tilde{\xi}, \tilde{\eta})$ for every $p \in M$, $\tilde{\xi}, \tilde{\eta} \in \tilde{E}$. More generally, (E, V, δ) is an *affine bundle with standard fiber* $(\tilde{E}, \tilde{V}, \tilde{\delta})$ iff each point of M is contained in an open subset $U \subseteq M$ such that $(\pi_E^{-1}(U), \pi_V^{-1}(U), (\delta|U))$ is isomorphic to the trivial affine bundle $(U \times \tilde{E}, U \times \tilde{V}, \tilde{\delta} \circ (\pi_2 \times \pi_2))$ defined above. Here the word isomorphism is understood in the technical sense of Definition 3.2 below. ∎

DEFINITION 3.2. Two affine bundles (E_1, V_1, δ_1) and (E_2, V_2, δ_2) over the same base manifold M are *isomorphic* iff there exists a pair of maps (φ_E, φ_V) such that

(1) $\varphi_E : E_1 \to E_2$ is a diffeomorphism such that $\varphi_E((E_1)_p) = (E_2)_p$ for each $p \in M$,

(2) $\varphi_V : V_1 \to V_2$ is a vector bundle isomorphism, and

(3) the following diagram is commutative:

$$\begin{array}{ccc} E_1 \times E_1 & \xrightarrow{\delta_1} & V_1 \\ \varphi_E \times \varphi_E \downarrow & & \downarrow \varphi_V \\ E_2 \times E_2 & \xrightarrow{\delta_2} & V_2 \end{array}$$

∎

DEFINITION 3.3. We say that (E, V, δ, σ) is a *pointed affine bundle* over M iff (E, V, δ) is an affine bundle over M and $\sigma : M \to E$ is a global section of E. If $(E_1, V_1, \delta_1, \sigma_1)$ and $(E_2, V_2, \delta_2, \sigma_2)$ are pointed affine bundles, then *they are said to be isomorphic* if (φ_E, φ_V) is an isomorphism from (E_1, V_1, δ_1) to (E_2, V_2, δ_2) such that $\varphi_E \circ \sigma_1 = \sigma_2$.

As an example of how such pointed affine bundles naturally arise, consider for the moment the special case when (V, M, π_V) is the tangent bundle of M. Recall that if LM is the frame bundle of M, then TM may be recovered as a bundle associated to LM and the usual action of $G\ell(m, \mathbf{R})$ on $\mathbf{R}^m$ via matrix multiplication (we assume M is m-dimensional). Indeed if $G\ell(m, \mathbf{R})$ acts on $LM \times \mathbf{R}^m$ via $g \cdot (u, v) = (ug^{-1}, gv)$ then the map φ_o from $E = (LM \times \mathbf{R}^m)/G\ell(m, \mathbf{R})$ into TM defined by $\varphi_o([u, v]) = v^i e_i$ is a vector bundle isomorphism (here $u = (p, e_i) \in LM$, $v \in \mathbf{R}^m$, and $[u, v]$ denotes the orbit of $G\ell(m, \mathbf{R})$ which contains (u, v)). Moreover the difference function field δ_o defined by $\delta_o([u, v], [u, w]) = (w^i - v^i)e_i$ provides us with an affine bundle (E, TM, δ_o) which implicitly arises from the map φ_o which identifies E and TM. In this case we see that the affine structure map δ_o arises from the canonical «soldering map» φ_o from E to M via $\delta_o(\eta, \xi) = \varphi_o(\xi) - \varphi_o(\eta)$. Our next proposition is trivial but it shows that every soldering map $\varphi : E \to TM$ gives rise to an affine structure δ on the pair (E, TM).

PROPOSITION 3.1. *Assume that (E, M, π_E) is a fiber bundle and that (V, M, π_V) is a vector bundle. If $\varphi : E \to V$ is a diffeomorphism such that $\varphi(E_p) = V_p$ for each $p \in M$, then the function δ_φ defined by $\delta_\varphi(\eta, \xi) = \varphi(\xi) - \varphi(\eta)$ for $\xi, \eta \in E_p$ and $p \in M$ is a difference function field such that (E, V, δ_φ) is an affine bundle. Moreover, if $\hat{0}$ is the zero section of V, then the map $\sigma_\varphi : M \to E$ defined by $\sigma_\varphi(p) = \varphi^{-1}(\hat{0}(p))$ is a global section of E and $(E, V, \delta_\varphi, \sigma_\varphi)$ is a pointed affine bundle. Conversely, if (E, V, δ) is an affine bundle and E admits a global section, then for every such section σ the map $\varphi_\sigma : E \to V$ defined by $\varphi_\sigma(\xi) = \delta_{\pi_E}(\sigma(\pi_E(\xi)), \xi)$ is a fiber preserving diffeomorphism such that $\delta_{\varphi_\sigma} = \delta$ and $\sigma_{\varphi_\sigma} = \sigma$.*

COROLLARY 3.2. *Every soldering map*

$$\varphi : \big((LM) \times \mathbf{R}^n\big)/G\ell(n,\mathbf{R}) \longrightarrow TM$$

defines an affine structure δ_φ. ∎

The proofs of these results are left to the reader.

Proposition 3.1 assures us that affine bundles exist in profusion.

For a given affine bundle (E, V, δ) we define the corresponding affine frame bundle $AE = A(E, V, \delta)$ to be the set of all elements (p, e_i, t) where $p \in M$, $\{e_i\}$ is a basis of V_p, and $t \in E_p$. We define an action of A_n on AE by

$$(p, e_i, t) \cdot (v, g) = (p, e_j g^j_i, \delta^{-1}_{\sigma(p)}(\delta_{\sigma(p)}(t) + v^i e_i))$$

where σ is any section of E and where $\delta_{\sigma(p)}$ is the mapping from E_p to V_p defined by $\xi \to \delta_p(\sigma(p), \xi)$. Clearly the action is independent of the choice of σ. Moreover it is easy to show that AE is a principal fiber bundle over M with group A_n and projection $\pi : AE \to M$ defined by $\pi(p, e_i, t) = p$.

Assume that the vector bundle V has fiber dimension n and that $(\tilde{E}, \tilde{V}, \tilde{\delta}, \tilde{\sigma})$ is any pointed affine space of dimension n. Since $\tilde{V}$ is a vector space of dimension n there is an obvious action of $G\ell(\tilde{V})$ on $\tilde{V}$ and this induces left actions of A_n on $\tilde{E}$ and $\tilde{V}$ as in Section 2 (see Equations (2.1) and (2.2)). Let $\overline{E}$ and $\overline{V}$ denote the fiber bundles associated to AE and the actions of A_n on $\tilde{E}$ and on $\tilde{V}$ respectively. In the next paragraph we show how to define maps $\overline{\delta}, \overline{\sigma}$ relative to which $(\overline{E}, \overline{V}, \overline{\delta}, \overline{\sigma})$ becomes a pointed affine bundle. We will then show that $(\overline{E}, \overline{V}, \overline{\delta}, \overline{\sigma})$ is isomorphic to (E, V, δ, σ).

We denote the associated bundles $\overline{E}$ and $\overline{V}$ by $\overline{E} = AE \times_{A_n} \tilde{E}$ and $\overline{V} = AE \times_{A_n} \tilde{V}$ respectively. Typical elements of $\overline{E}$ will be denoted by

$$[u, \tilde{\xi}] = \{(ua^{-1}, a\tilde{\xi}) | a \in A_n\}$$

for $(u, \tilde{\xi}) \in AE \times \tilde{E}$. Analogous notation is used for $\overline{V}$. We define a difference function field $\overline{\delta} : \overline{E} \times \overline{E} \to \overline{V}$ by

$$\overline{\delta}_p([u, \tilde{\eta}], [u, \tilde{\xi}]) = [u, \tilde{\delta}(\tilde{\eta}, \tilde{\xi})]$$

for $u \in AE$, $\tilde{\eta}, \tilde{\xi} \in \tilde{E}$. It is easy to check that $\overline{\delta}$ is well-defined precisely when $a \cdot \tilde{\delta} = \tilde{\delta}$ for $a \in A_n$. On the other hand the actions of A_n on $\tilde{E}$ and on $\tilde{V}$ were defined in terms of $\tilde{\delta}$ in such a way that $a \cdot \tilde{\delta} = \tilde{\delta}$ for all $a \in A_n$:

$$\begin{aligned}
(a \cdot \tilde{\delta})(\tilde{\eta}, \tilde{\xi}) &= g\tilde{\delta}(a^{-1} \cdot \tilde{\eta}, a^{-1} \cdot \tilde{\xi}) \\
&= g\tilde{\delta}\big(\tilde{\delta}^{-1}_{\tilde{\sigma}}(g^{-1}(\tilde{\delta}_{\tilde{\sigma}}(\tilde{\eta}) - v)),\ \tilde{\delta}^{-1}_{\tilde{\sigma}}(g^{-1}(\tilde{\delta}_{\tilde{\sigma}}(\tilde{\xi} - v))\big) \\
&= g[g^{-1}(\tilde{\delta}_{\tilde{\sigma}}(\tilde{\xi}) - v) - g^{-1}(\tilde{\delta}_{\tilde{\sigma}}(\tilde{\eta}) - v)] \\
&= \tilde{\delta}_{\tilde{\sigma}}(\tilde{\xi}) - \tilde{\delta}_{\tilde{\sigma}}(\tilde{\eta}) = \tilde{\delta}(\tilde{\eta}, \tilde{\xi})
\end{aligned} \tag{3.1}$$

(here $a = (v,g) \in A_n$, $(\tilde{\eta},\tilde{\xi}) \in \tilde{E} \times \tilde{E}$). Note that if σ is a global section of E then we may define a mapping $\overline{\sigma} : M \to \overline{E}$ by $\overline{\sigma}(p) = [(p, e_i, \sigma(p)), \tilde{\sigma}]$. Observe that $\overline{\sigma}$ is well-defined as it depends only on $\tilde{\sigma}$ and σ and not on the choice of frame $\{e_i\}$.

With these definitions $(\overline{E}, \overline{V}, \overline{\delta})$ is an affine bundle which is pointed when (E, V, δ) is pointed.

PROPOSITION 3.3. *The pointed affine bundles* $(\overline{E}, \overline{V}, \overline{\delta}, \overline{\sigma})$ *and* (E, V, δ, σ) *are isomorphic.*

Proof. Define maps $\varphi_E : \overline{E} \to E$ and $\varphi_V : \overline{V} \to V$ by

$$\varphi_E([(p, e_i, t), \tilde{\xi}]) = \delta^{-1}_{\sigma(p)}((\tilde{\xi}^i + t^i)e_i) \tag{3.2}$$

and

$$\varphi_V([(p, e_i, t), \tilde{w}]) = \tilde{w}^i e_i \tag{3.3}$$

where $\delta_{\sigma(p)}(t) = t^i e_i$, $\tilde{\delta}_{\tilde{\sigma}}(\tilde{\xi}) = \tilde{\xi}^i r_i$, $\tilde{w} = \tilde{w}^i r_i$ and $\{r_i\}$ is some fixed basis of $\tilde{V}$. To show that (φ_E, φ_V) is an isomorphism it is sufficient to restrict one's attention to elements of AE of the form $(p, e_i, \sigma(p))$ since typical elements of AE such as (p, e_i, t) may be written as $(p, e_i, \sigma(p)) \cdot a$ for some $a \in A_n$ and since

$$\overline{E} = \{[(p, e_i, \sigma(p)), \tilde{\xi}] | (p, e_i) \in LM, \tilde{\xi} \in \tilde{E}\}$$
$$\overline{V} = \{[(p, e_i, \sigma(p)), \tilde{w}] | (p, e_i) \in LM, \tilde{w} \in \tilde{V}\}.$$

In this notation $\varphi_E([(p, e_i, \sigma(p)), \tilde{\xi}]) = \delta^{-1}_{\sigma(p)}(\tilde{\xi}^i e_i)$ and $\varphi_V([(p, e_i, \sigma(p)), \tilde{w}]) = \tilde{w}^i e_i$ where $\tilde{\delta}_{\tilde{\sigma}}(\tilde{\xi}) = \tilde{\xi}^i r_i$ and $\tilde{w} = \tilde{w}^i r_i$. It is immediate that both φ_E and φ_V map the fibers $\overline{E}_p, \overline{V}_p$ diffeomorphically onto the fibers E_p, V_p respectively. Moreover it is obvious that $\varphi_V | \overline{V}_p$ is linear. It is easy to check that both maps are smooth maps as this follows from the analogous arguments in the linear frame bundle case. To check (3) of Definition 3.1 observe that

$$\begin{aligned}
&\delta_p(\varphi_E([(p, e_i, \sigma(p)), \tilde{\xi}_1]),\ \varphi_E([(p, e_i, \sigma(p)), \tilde{\xi}_2])) \\
&= \delta_p(\delta^{-1}_{\sigma(p)}(\tilde{\xi}^i_1 e_i), \delta^{-1}_{\sigma(p)}(\tilde{\xi}^i_2 e_i)) \\
&= (\tilde{\xi}^i_2 - \tilde{\xi}^i_1)e_i \\
&= \tilde{\delta}^i(\tilde{\xi}_1, \tilde{\xi}_2)e_i \\
&= \varphi_V([(p, e_i, \sigma(p)), \tilde{\delta}(\tilde{\xi}_1, \tilde{\xi}_2)]) \\
&= \varphi_V(\overline{\delta}_p([(p, e_i, \sigma(p)), \tilde{\xi}_1],\ [(p, e_i, \sigma(p)), \tilde{\xi}_2]))
\end{aligned}$$

where $\tilde{\delta}(\tilde{\xi}_1, \tilde{\xi}_2) = \tilde{\delta}^i(\tilde{\xi}_1, \tilde{\xi}_2) r_i$. It is obvious that $(\varphi_E \circ \overline{\sigma})(p) = \sigma(p)$ and the proposition follows.

It follows that (E, V, δ, σ) may be identified with $(\overline{E}, \overline{V}, \overline{\delta}, \overline{\sigma})$ via (φ_E, φ_V). We refer to the inverse of this pair (φ_E, φ_V) as the *standard soldering* of (E, V, δ, σ) onto

$$(\overline{E}, \overline{V}, \overline{\delta}, \overline{\sigma}) .$$

This terminology arises from the fact that in the linear theory if we let $V = TM$ be the tangent bundle and $\varphi_V : \overline{V} \to V = TM$ be the standard identification of $\overline{V} = LM \times_{G\ell(n)} \mathbb{R}^n$ with V, then the soldering form Θ on LM is defined via φ_V^{-1} by

$$\Theta_u(X) = u^{-1}(\varphi_V^{-1}(d_u\pi(X)))$$

for $u \in LM$, $X \in T_u(LM)$. The map φ_V^{-1} is usually suppressed but it is actually TM which is being soldered via φ_V^{-1} to the associated bundle $LM \times_{G\ell(n,\mathbb{R})} \mathbb{R}^n$. In the affine case one defines $E = TM$ and $V = TM$. For each p, $\delta_p : E_p \times E_p \to V_p$ is defined by $\delta_p(v, w) = w - v$. It follows that (E, V, δ, σ) is a pointed affine bundle if we define $\sigma(p)$ to be the zero vector in T_pM for each $p \in M$. Then $(\varphi_E^{-1}, \varphi_V^{-1})$ solders (TM, TM, δ, σ) to the corresponding associated bundle $(\overline{E}, \overline{V}, \overline{\delta}, \overline{\sigma})$.

We are now prepared to introduce the vector bundle whose sections are our principal object of study. For a given affine bundle (E, V, δ), let $\mathcal{V}^2(E, V)$ be the union $\bigcup_{p\in M} \mathcal{V}_s^2(E_p, V_p)$ where $\mathcal{V}_s^2(E_p, V_p)$ is defined as in Section 2 relative to the affine structure of δ_p. Each of the vector spaces $\mathcal{V}_s^2(E_p, V_p)$ is infinite dimensional, but given some reasonable topology on each of them $\mathcal{V}^2(E, V)$ will acquire a vector bundle structure. Clearly every difference function field on E defines a section of this bundle and these fields are the carriers of affine structure.

If $\tilde{E}$ and $\tilde{V}$ are the standard fibers of E and V, respectively, then $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$ is the standard fiber of $\mathcal{V}^2(E, V)$, thus it suffices to find a topology on $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$ subject to the conditions:

(T1) $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$ must be a Fréchet space,

(T2) the mapping from $\mathcal{V}_s^2(\tilde{E}, \tilde{V}) \times \tilde{E}^2$ to $\tilde{V}$ defined by $(\alpha, x) \mapsto \alpha(x)$ must be smooth,

(T3) the action of A_n on $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$ defined by equation (2.1) must be smooth.

To see that such a topology exists on $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$, first observe that $C^\infty(\tilde{E}^2, \tilde{V})$ may be identified with $C^\infty(\mathbb{R}^{2n}, \mathbb{R})$. If we give $C^\infty(\mathbb{R}^{2n}, \mathbb{R})$ the weakest topology which implies uniform convergence of sequences of functions and their derivatives on compacta, then $C^\infty(\mathbb{R}^{2n}, \mathbb{R})$, and thus $C^\infty(\tilde{E}^2, \tilde{V})$, is a Fréchet space. This topology is called the Schwartz C^∞-topology and although Rudin does not refer to it by name he discusses it and its properties in detail in [14] (see pages 33, 137). Since $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$ is a closed subspace of $C^\infty(\tilde{E}^2, \tilde{V})$, it is a Fréchet space as well. Thus $\mathcal{V}^2_s(\tilde{E}, \tilde{V})$ satisfies condition (T1) above. Straightforward, but lengthy, arguments may be constructed which show that (T2) and (T3) also hold. The author is indebted to the referee for the observation that such arguments can be circumvented by noting that Fréchet spaces are convenient vector spaces in the sense of [4] and that (T2) and (T3) follow from the general results established for convenient vector spaces in [4]. These remarks prove part (1) of the following theorem. ∎

THEOREM 3.4. *If (E, V, δ) is an affine bundle over a manifold M with standard fiber $(\tilde{E}, \tilde{V}, \tilde{\delta})$, then*

(1) *$\mathcal{V}^2_s(\tilde{E}, \tilde{V})$ is a Fréchet space relative to the Schwartz C^∞-topology described above and $\mathcal{V}^2_s(\tilde{E}, \tilde{V})$ satisfies (T1), (T2), (T3) relative to this topology,*

(2) *$\mathcal{V}^2(E, V) = \bigcup_{p\in M} \mathcal{V}^2_s(E_p, V_p)$ is a vector bundle with standard fiber $\mathcal{V}^2_s(\tilde{E}, \tilde{V})$, and*

(3) *for arbitrary smooth (local) sections $\sigma_1, \sigma_2 : U \to E$ of E and $\alpha : U \to \mathcal{V}^2(E, V)$ of $\mathcal{V}^2(E, V)$ the mapping defined by $p \mapsto \alpha(p)(\sigma_1(p), \sigma_2(p))$ is a smooth section of V.*

Proof. The proof of statement (1) was outlined prior to the statement of the theorem. The proof of (2) requires the development of some ideas which are useful below. Consequently we defer the proof of (2) preferring to first prove (3) using (2). Given local sections $\sigma_1, \sigma_2 : U \to E$ and $\alpha : U \to \mathcal{V}^2(E, V)$, it is obvious that the mapping $\alpha(\sigma_1, \sigma_2)$ defined by $p \mapsto \alpha(p)(\sigma_1(p), \sigma_2(p))$ is a section of the vector bundle V. We have only to show that $\alpha(\sigma_1, \sigma_2)$ is smooth. Since smoothness is a local property, it suffices to assume that E, V and $\mathcal{V}^2(E, V)$ are trivial over U with trivializing mappings ψ_U, φ_U and Γ_U, respectively. It is shown in the proof of (2) below that Γ_U may be defined to be the mapping from $\bigcup_{p\in U} \mathcal{V}^2_s(E_p, V_p)$ to $U \times \mathcal{V}^2_s(\tilde{E}, \tilde{V})$ given by $\Gamma_U(\beta) = (p, \tilde{\beta}_U)$ where $\beta \in \mathcal{V}^2_s(E_p, V_p)$ and where

$$\tilde{\beta}_U(\tilde{\eta}, \tilde{\xi}) = \pi_2(\varphi_U(\beta(\psi_U^{-1}(p, \tilde{\eta}), \psi_U^{-1}(p, \tilde{\xi}))))$$

for $(\tilde{\eta}, \tilde{\xi}) \in \tilde{E} \times \tilde{E}$. If we define $\tilde{\xi}_i : U \to \tilde{E}$, $i = 1, 2$, by $(p, \tilde{\xi}_i(p)) = \psi_U(\sigma_i(p))$ for all $p \in U$, then clearly $\tilde{\xi}_1, \tilde{\xi}_2$ are smooth and, for each $p \in U$,

$$\alpha(\sigma_1, \sigma_2)(p) = \varphi_U^{-1}(p, \widetilde{\alpha(p)}_U(\tilde{\xi}_1(p), \tilde{\xi}_2(p))) .$$

The mapping $p \mapsto \widetilde{\alpha(p)}_U$ is smooth since $\widetilde{\alpha(p)}_U = \pi_2(\Gamma_U(\alpha(p)))$. It follows from the topological property (T2) above that the mapping $p \mapsto \widetilde{\alpha(p)}_U(\tilde{\xi}_1(p), \tilde{\xi}_2(p))$ is smooth. Thus $\alpha(\sigma_1, \sigma_2)$ is smooth and (3) follows.

To prove (2) it is necessary to show that there exists a family $\{\Gamma_U\}$ of trivializing mappings whose transition functions are smooth. It turns out to be useful to know that not only is this true but, in fact, these transition functions are related to transition functions of AE in such a way that one can show that the vector bundle $\mathcal{V}^2(E,V)$ is an associated bundle of AE. Since we need this fact in Sections 4 and 5 we proceed to develop these ideas in more detail.

Assume that we are given an open cover $\{U\}$ of M and, for each U in the cover, a pair of trivializing mappings

$$\psi_U : \pi_E^{-1}(U) \to U \times \tilde{E} \qquad \varphi_U : \pi_V^{-1}(U) \to U \times \tilde{V}$$

which satisfy the third condition of Definition 3.2. For U and W in the cover such that $U \cap W \neq \emptyset$, let $f_{UW} : U \cap W \to \mathrm{Diff}(\tilde{E})$ and $g_{UW} : U \cap W \to G\ell(\tilde{V})$ be defined by

$$\text{(3.4)} \qquad \begin{aligned} (\psi_W \circ \psi_U^{-1})(p,\tilde{\xi}) &= (p, f_{UW}(p)(\tilde{\xi})) \\ (\varphi_W \circ \varphi_U^{-1})(p,\zeta) &= (p, g_{UW}(p)(\zeta)) \end{aligned}$$

for $p \in U \cap W$ and $\tilde{\xi} \in \tilde{E}$, $\zeta \in \tilde{V}$. We have assumed that the maps $\{\varphi_U\}$ are all linear on fibers and it can be shown using (3) of Definition 3.2 that the mappings $f_{UW}(p)$ are affine mappings for each $p \in U \cap W$. Indeed an easy computation shows that

$$\tilde{\delta}(f_{UW}(p)(\tilde{\xi}), f_{UW}(p)(\tilde{\eta})) = g_{UW}(p)(\tilde{\delta}(\tilde{\xi},\tilde{\eta}))$$

for $p \in U \cap W$, $\tilde{\xi},\tilde{\eta} \in \tilde{E}$. This fact may be used to show that

$$\text{(3.5)} \qquad f_{UW}(p)(\tilde{\xi}) = \tilde{\delta}_{t_0}^{-1}(g_{UW}(p)\tilde{\delta}_{t_0}(\tilde{\xi}) + v(p))$$

where $p \in U \cap W$, $\tilde{\xi} \in \tilde{E}$, $t_0 \in \tilde{E}$, and $v(p) = \tilde{\delta}_{t_0}(f_{UW}(p)(t_0))$. Given $t_0 \in \tilde{E}$ and a fixed basis $\{\tilde{e}_i\}$ of $\tilde{V}$ we define local sections $\{\tau_U\}$ of AE by $\tau_U(p) = (p, \varphi_U^{-1}(p,\tilde{e}_i),\ \psi_U^{-1}(p,t_0))$ for $p \in U$. If U and W are members of the given cover of M such that $U \cap W \neq \emptyset$ then τ_U and τ_W are related by the identity

$$\text{(3.6)} \qquad \tau_U(p) = \tau_W(p)a_{UW}(p)$$

where $p \in U \cap W$ and $a_{UW}(p) = (v(p), g_{UW}(p)) \in V \rtimes G\ell(V)$. Thus $\{a_{UW}\}$ is a set of cocycles of AE, the cocycles determined by the local trivialization of (E,V,δ).

Using the trivializations $\{\psi_U\}$ and $\{\varphi_U\}$ we define mappings $\{\Gamma_U\}$ as follows. For each U in the cover, let Γ_U be the mapping from $\mathcal{V}^2(\pi_E^{-1}(U), \pi_V^{-1}(U))$ into

$U \times \mathcal{V}^2_s(\tilde{E}, \tilde{V})$ defined by $\Gamma_U(\alpha) = (\pi(\alpha), \tilde{\alpha}_U)$, where $\pi : \mathcal{V}^2(E, V) \to M$ is given by requiring that $\pi(\mathcal{V}^2(E_p, V_p)) = p$ for every $p \in M$ and where

$$\tilde{\alpha}_U(\tilde{\xi}, \tilde{\eta}) = \pi_2(\varphi_U(\alpha(\psi_U^{-1}(\pi(\alpha), \tilde{\xi}),\ \psi_U^{-1}(\pi(\alpha), \tilde{\eta}))))$$

for $\tilde{\xi}, \tilde{\eta} \in \tilde{E}$. The mapping Γ_U is a bijection and is linear on fibers. Moreover, for $p \in U \cap W,\ \alpha \in \pi^{-1}(\{p\})$, we have

$$\begin{aligned}\tilde{\alpha}_W(\tilde{\xi}, \tilde{\eta}) &= \pi_2(\varphi_W(\alpha(\psi_W^{-1}(p, \tilde{\xi}), \psi_W^{-1}(p, \tilde{\eta})))) \\ &= g_{UW}(p)(\pi_2((\varphi_U \circ \alpha \circ (\psi_U^{-1} \times \psi_U^{-1})) \\ &\quad ((p, f_{WU}(p)(\tilde{\xi})), (p, f_{WU}(p)(\tilde{\eta}))))) \\ &= g_{UW}(p)(\tilde{\alpha}_U(f_{WU}(p)(\tilde{\xi}), f_{WU}(p)(\tilde{\eta}))\end{aligned}$$

for $\tilde{\xi}, \tilde{\eta} \in \tilde{E}$. If we define maps $\{h_{UW}\}$ by

$$h_{UW}(p)(\beta) = g_{UW}(p) \circ \beta \circ (f_{UW}(p)^{-1} \times f_{UW}(p)^{-1}) \tag{3.7}$$

for $p \in U \cap W$ and $\beta \in \pi^{-1}(\{p\})$, then

$$\Gamma_W(\alpha) = (\pi(\alpha), h_{UW}(p)(\pi_2(\Gamma_U(\alpha))))$$

and

$$(\Gamma_W \circ \Gamma_U^{-1})(p, \tilde{\alpha}) = (p, h_{UW}(p)(\tilde{\alpha})) \tag{3.8}$$

for $p \in U \cap W,\ \alpha \in \pi^{-1}(\{p\}),\ \tilde{\alpha} \in \mathcal{V}^2_s(\tilde{E}, \tilde{V})$. Thus $\{h_{UW}\}$ is a set of transition functions for $\mathcal{V}^2(E, V)$ and it will follow that $\mathcal{V}^2(E, V)$ is a vector bundle if we can show that the $\{h_{UW}\}$ are smooth.

It is obvious that g_{UW} and f_{UW} are smooth and consequently so is the mapping $\Phi_1 : (U \cap W) \times \mathcal{V}^2 \to G\ell(V) \times \mathcal{V}^2 \times \mathrm{Aff}(\tilde{E}^2)$ defined by $\Phi_1(p, \tilde{\alpha}) = (g_{UW}(p), \tilde{\alpha}, f_{UW}(p)^{-1} \times f_{UW}(p)^{-1})$. Moreover the mapping $\Phi_2 : G\ell(V) \times \mathcal{V}^2 \times \mathrm{Aff}(\tilde{E}^2) \to \mathcal{V}^2$ defined by $\Phi_2(g, \tilde{\alpha}, \tilde{f}) = g \circ \tilde{\alpha} \circ \tilde{f}$ is also smooth (see [4], [6], or [12]). Since $h_{UW}(p)(\tilde{\alpha}) = \Phi_2(\Phi_1(p, \tilde{\alpha}))$, for $p \in U \cap W,\ \tilde{\alpha} \in \mathcal{V}^2_s(\tilde{E}, \tilde{V})$ it follows that $(p, \tilde{\alpha}) \mapsto h_{UW}(p)(\tilde{\alpha})$ is a smooth mapping. Many authors consider this to be adequate in order that $\mathcal{V}^2(E, V)$ be a vector bundle (see [6] or [12]). Others require that $h_{UW} : U \cap W \to \mathrm{End}(\mathcal{V}^2_s(\tilde{E}, \tilde{V}))$ be smooth. Since $\mathcal{V}^2_s(\tilde{E}, \tilde{V})$ is a Fréchet space, it is a convenient vector space in the sense of [4]. It follows from Theorem 3.6.5 of [4] that End $(\mathcal{V}^2_s(\tilde{E}, \tilde{V}))$ has a convenient structure relative to which the mappings $\{h_{UW}\}$ are smooth. Thus $\mathcal{V}^2(E, V)$ is a vector bundle in this stronger sense as well. The proof of Theorem 3.4 is now complete.

Observe that in the process of proving Theorem 3.4 we have shown that the mappings $\{a_{UW}\}$ defined by (3.6) is a set of cocycles for the affine frame bundle AE, and that $\{h_{UW}\}$ is a set of cocycles for the vector bundle $\mathcal{V}^2(E,V)$ (see equation (3.8)). Moreover, equation (3.5) shows that $f_{UW}(p)$ is the affine mapping on $\tilde{E}$ which is identified with $(v(p), g_{UW}(p)) = a_{UW}(p)$ relative to the splitting of $\mathrm{Aff}(\delta_0)$ as a semidirect product $V \rtimes G\ell(V)$ defined in Section 2. Thus we may identify the mappings $\{f_{UW}\}$ as a set of cocycles of AE, and equation (3.7) shows that the $\{f_{UW}\}$ and $\{h_{UW}\}$ are related by the identity

$$h_{UW}(p)(\beta) = f_{UW}(p) \cdot \beta \tag{3.9}$$

where $p \in U \cap W$ and where $f_{UW}(p) \cdot \beta$ denotes the action of $f_{UW}(p) \in \mathrm{Aff}(\delta_0)$ on $\beta \in \mathcal{V}^2_s(\tilde{E},\tilde{V})$ defined by equation (2.1). A local trivialization for the vector bundle $AE \times_{A_n} \mathcal{V}^2_s(\tilde{E},\tilde{V})$ is given by the family of mappings $\{\tilde{\Gamma}_U\}$ defined by requiring that

$$\tilde{\Gamma}_U([\tau_U(p), \tilde{\alpha}]) = (p, \tilde{\alpha})$$

for $p \in U$, $\tilde{\alpha} \in \mathcal{V}^2_s(\tilde{E},\tilde{V})$. It follows easily from (3.9) that the $\{h_{UW}\}$ are transition functions for this trivialization. Moreover, general arguments such as those in Husemoller [8], pages 59-64, and Vaisman [16], page 106, show that $\mathcal{V}^2(E,V)$ is isomorphic to the bundle associated to the affine frame bundle AE and the action defined by equation (2.1). Neither of the cited references formulate their arguments in the category of Fréchet vector bundles, but Frölicher and Kriegl [4] provide us with a setting general enough that the usual arguments may be used to show that $\mathcal{V}^2(E,V)$ is isomorphic to the associated bundle $AE \times_{\mathrm{Aff}(\delta_0)} \mathcal{V}^2_s(\tilde{E},\tilde{V})$. ∎

COROLLARY 3.5. *The vector bundle $\mathcal{V}^2(E,V)$ is isomorphic to the bundle associated to the principal bundle of affine frames AE of (E,V,δ) and the action of $V \rtimes G\ell(V)$ on the Fréchet space $\mathcal{V}^2_s(\tilde{E},\tilde{V})$ defined by equation (2.1).* ∎

Although the cocycle argument given above proves Corollary 3.5, we find it useful to have an explicit formula for this isomorphism in Sections 4 and 5. For $(u,\alpha) \in AE \times \mathcal{V}^2_s(\tilde{E},\tilde{V})$, let $(u,\alpha)^\star$ denote the element of $\mathcal{V}^2(\overline{E},\overline{V})$ defined by

$$(u,\alpha)^\star_p([u,\tilde{\eta}],[u,\tilde{\xi}]) = [u, \alpha(\tilde{\eta},\tilde{\xi})] \tag{3.10}$$

where $u \in AE$, $\tilde{\eta},\tilde{\xi} \in \tilde{E}$ and $\pi(u) = p$. Here $\overline{E} = AE \times_{A_n} \tilde{E}$, $\overline{V} = AE \times_{A_n} \tilde{V}$, and $\mathcal{V}^2(E,V)$ is identified with $\mathcal{V}^2(\overline{E},\overline{V})$. The mapping $(u,\alpha) \mapsto (u,\alpha)^\star$ is constant on orbits of the action of A_n on $AE \times \mathcal{V}^2_s(\tilde{E},\tilde{V})$ and consequently defines a mapping from $AE \times_{A_n} \mathcal{V}^2_s(\tilde{E},\tilde{V}) := [AE \times \mathcal{V}^2_s(\tilde{E},\tilde{V})]/A_n$ onto $\mathcal{V}^2(\overline{E},\overline{V})$. A straightforward, but tedious argument, shows that the mapping defined by

$$[u,\alpha] \mapsto (u,\alpha)^\star \tag{3.11}$$

is an explicit isomorphism from $AE \times_{A_n} \mathcal{V}^2_s(\tilde{E},\tilde{V})$ onto $\mathcal{V}^2(\overline{E},\overline{V})$.

With Corollary 3.5 in hand, it is trivial to obtain the usual one-to-one correspondence between sections of $\mathcal{V}^2(E,V)$ and equivariant maps from AE into $\mathcal{V}^2(\tilde{E},\tilde{V})$. Given a section $\sigma : M \to \mathcal{V}^2(E,V) \cong AE \times_{A_n} \mathcal{V}^2_s(\tilde{E},\tilde{V})$ the corresponding equivariant mapping is the unique equivariant mapping $\hat{\sigma} : AE \to \mathcal{V}^2_s(\tilde{E},\tilde{V})$ such that

$$\sigma(p) = [u, \hat{\sigma}(u)] \tag{3.12}$$

for any $u \in AE$ such that $\pi(u) = p$. Conversely, given any equivariant mapping $\hat{\sigma} : AE \to \mathcal{V}^2_s(\tilde{E},\tilde{V})$, equation (3.7) clearly defines a unique section σ of $\mathcal{V}^2(E,V)$.

PROPOSITION 3.6. *Assume $\sigma, \tau : U \to E$ are arbitrary local sections of E and that $\alpha : U \to \mathcal{V}^2(E,V)$ is any local section of $\mathcal{V}^2(E,V)$. Let $\hat{\sigma}, \hat{\tau}, \hat{\alpha}$ be the corresponding equivariant mappings from $(AE)|_U$ into $\tilde{E}, \tilde{E}$, and $\mathcal{V}^2(\tilde{E},\tilde{V})$, respectively. Then $\hat{\alpha}(\hat{\sigma},\hat{\tau})$ is an equivariant mapping from $(AE)|_U$ into $\tilde{V}$ and the corresponding section of V is $\alpha(\sigma,\tau)$.*

Proof. For $u \in AE$ and $a \in A_n$ we have

$$\begin{aligned}
\hat{\alpha}(\hat{\sigma},\hat{\tau})(ua) &:= \hat{\alpha}(ua)(\hat{\sigma}(ua), \hat{\tau}(ua)) \\
&= (a^{-1}\cdot\hat{\alpha})(u)(a^{-1}\cdot\hat{\sigma}(u),\ a^{-1}\cdot\hat{\tau}(u)) \\
&= g^{-1}\hat{\alpha}(u)(aa^{-1}\hat{\sigma}(u),\ aa^{-1}\hat{\tau}(u)) \\
&= g^{-1}\hat{\alpha}(\hat{\sigma},\hat{\tau})(u)
\end{aligned}$$

where $a = (v,g) \in A_n$. Thus $\hat{\alpha}(\hat{\sigma},\hat{\tau})$ is an equivariant mapping from AE into $\tilde{V}$.

To establish the correspondence between $\alpha(\sigma,\tau)$ and $\hat{\alpha}(\hat{\sigma},\hat{\tau})$ we find it convenient to pass to associated bundle notation: $E \cong \overline{E}$, $V \cong \overline{V}$, $\mathcal{V}^2(E,V) \cong \mathcal{V}^2(\overline{E},\overline{V}) \cong AE \times_{A_n} \mathcal{V}^2_s(\tilde{E},\tilde{V})$. Then

$$\begin{aligned}
\alpha(\sigma,\tau)(p) &= \alpha(p)(\sigma(p),\tau(p)) \\
&= [u,\hat{\alpha}(u)]([u,\hat{\sigma}(u)],[u,\hat{\tau}(u)]) \\
&= (u,\hat{\alpha}(u))^*([u,\hat{\sigma}(u)],[u,\hat{\tau}(u)]) \\
&= [u,\hat{\alpha}(u)(\hat{\sigma}(u),\hat{\tau}(u))] \\
&= [u,\hat{\alpha}(\hat{\sigma},\hat{\tau})(u)]
\end{aligned}$$

for any $u \in AE$ such that $\pi(u) = p$. The proposition follows. ∎

4. COVARIANT DERIVATIVES OF DIFFERENCE FUNCTION FIELDS AND THEIR GEOMETRICAL INTERPRETATION

In this section we develop fundamental facts about the covariant derivatives of difference function fields defined on a given affine bundle (E, V, δ). To do this we first consider the exterior covariant derivative of equivariant maps $\hat{\alpha}$ from AE into $\mathcal{V}^2(\tilde{E}, \tilde{V})$. This derivative will lead us to corresponding facts regarding the covariant derivative of sections of $\mathcal{V}^2(\overline{E}, \overline{V}) \cong \mathcal{V}^2(E, V)$. This procedure is analogous to a corresponding development for metrics whereby exterior covariant derivatives of equivariant maps $\hat{g} : LM \to T_2^0\mathbb{R}^n$ lead to relevant facts regarding the covariant derivative $\nabla_X g$ of metrics g defined on a manifold M. In this section we will also discuss the extent to which an equivariant mapping $\hat{\delta}$ defined by a difference function field δ may be regarded as a symmetry-breaking field. In such a case we will see that a connection ω on AE reduces to a subbundle $\hat{\delta}^{-1}(\delta_o)$ iff $D^\omega\hat{\delta} = 0$. This development is again analogous to what happens in the metric case whereby a connection ω on the frame bundle LM reduces to the bundle of orthonormal frames iff $D^\omega\hat{g} = 0$. There are important differences, however, as it is not clear under what circumstances a difference function field induces a symmetry-breaking equivariant mapping $\hat{\delta}$. We are able to analyze the situation in the special case that $\hat{\delta}$ has its values in the space $\mathrm{Aff}^2(\tilde{E}, \tilde{V}) \cap \mathcal{D}_s(\tilde{E}, \tilde{V})$ of affine difference function fields.

We first show how to obtain the exterior covariant derivative of an arbitrary smooth equivariant mapping $\hat{\alpha}$ from AE into $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$. This is given by the usual formula once one has a definition of the exterior derivative. The way has been paved for this in Section 3. Even though $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$ is infinite dimensional we need only the topological conditions (T1) - (T3) to guarantee that for $u \in AE$ and $X \in T_u(AE)$ we may define $d_u\hat{\alpha}(X)$ by the formula

$$(d_u\hat{\alpha})(X) = \frac{d}{dt}(\hat{\alpha} \circ \gamma)(0)$$

where $\gamma : (-\epsilon, \epsilon) \to AE$ is any curve such that $\gamma(0) = u$, $\frac{d\gamma}{dt}(0) = X$. Moreover, it follows easily that

$$(d_u\hat{\alpha})(X)(\tilde{\eta}, \tilde{\xi}) = d_u(ev_{(\tilde{\eta},\tilde{\xi})} \circ \hat{\alpha})(X)$$

where $ev_{(\tilde{\eta},\tilde{\xi})} \circ \hat{\alpha}$ is a function from AE into the finite dimensional vector space $\tilde{V}$. In particular we have

$$ev_{(\tilde{\eta},\tilde{\xi})} \circ d\hat{\alpha} = d(ev_{(\tilde{\eta},\tilde{\xi})} \circ \hat{\alpha}) \tag{4.1}$$

DEFINITION 4.1. If ω is a connection on the principal fiber bundle AE and $\hat{\alpha} : AE \to \mathcal{V}_s^2(\tilde{E}, \tilde{V})$ is an equivariant mapping, then *the exterior covariant derivative of* $\hat{\alpha}$, denoted by $D^\omega \hat{\alpha} = D\hat{\alpha}$, is the mapping from TAE into $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$ defined by $D_u \hat{\alpha}(X) = d_u \hat{\alpha}(hor\ X)$ for $u \in AE$, $X \in T_u AE$. Here hor X denotes the horizontal part of the tangent vector X (see [10]).

Since $\hat{\alpha} : AE \to \mathcal{V}_s^2(\tilde{E}, \tilde{V})$ has its values in a vector space the standard argument applies to show that one still has the identity

$$(D_u \hat{\alpha})(X) = d_u \hat{\alpha}(X) + \omega(X) \cdot \hat{\alpha}(u) \tag{4.1}$$

for $u \in AE$, $X \in T_u AE$. Here $\omega(X) \cdot \hat{\alpha}(u)$ denotes the action of the Lie algebra element $\omega(X) \in a_n$ on $\hat{\alpha}(u) \in \mathcal{V}_s^2(\tilde{E}, \tilde{V})$ explicated in Section 2. In fact it follows from the definition of this that

$$\begin{aligned}(D_u \hat{\alpha})(X)(\tilde{\eta}, \tilde{\xi}) =& (d_u \hat{\alpha})(X)(\tilde{\eta}, \tilde{\xi}) + \omega_L(X) \cdot \hat{\alpha}(u)(\tilde{\eta}, \tilde{\xi}) \\ & - d_{(\tilde{\eta}, \tilde{\xi})}(\hat{\alpha}(u))(\omega(X) \cdot \tilde{\eta},\ \omega(X) \cdot \tilde{\xi})\end{aligned} \tag{4.2}$$

where $u \in AE$, $X \in T_u(AE)$ and $\tilde{\eta}, \tilde{\xi} \in \tilde{E}$. Here $\omega_L(X)$ denotes the «linear part» of $\omega(X)$ and the term $d_{(\tilde{\eta}, \tilde{\xi})}(\hat{\alpha}(u))$ is indicative of the nonlinearity implicit in elements of $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$.

When one considers the fact that equivariant maps of the type $\hat{\alpha}$ arise from general diffeomorphisms, it is somewhat surprising that all the nonlinear features of $D\hat{\alpha}$ may be encapsulated into the one term $d_{(\tilde{\eta}, \tilde{\xi})}(\hat{\alpha}(u))(\omega(X)\tilde{\eta},\ \omega(X)\tilde{\xi})$. On the other hand this term is in general not easily simplified (although the bilinearity of $d(\hat{\alpha}(u))$ is of some use in this respect).

In the special case that $\hat{\alpha}$ has all of its values in the set $\mathrm{Aff}^2(\tilde{E}, \tilde{V})$ of affine elements of $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$ some reduction of (4.2) is possible. In more detail, recall from Section 2 that if $\hat{\alpha}(u) \in \mathrm{Aff}^2(\tilde{E}, \tilde{V})$ then there exist linear maps $\ell_1(u)$, $\ell_2(u)$ from $\tilde{V}$ to $\tilde{V}$ and a vector $v(u) \in \tilde{V}$ such that

$$\hat{\alpha}(u) = \ell_2(u) \circ \tilde{\delta}_{\tilde{\sigma}} \circ \pi_2 - \ell_1(u) \circ \tilde{\delta}_{\tilde{\sigma}} \circ \pi_1 + v(u)\ .$$

Since $v(u) = \hat{\alpha}(u)(\tilde{\sigma}, \tilde{\sigma})$, $\ell_2(u)(x) = \hat{\alpha}(u)(\tilde{\sigma}, \tilde{\delta}_{\tilde{\sigma}}^{-1}(x)) - v(u)$, and $\ell_1(u)(x) = -\hat{\alpha}(u)(\tilde{\delta}_{\tilde{\sigma}}^{-1}(x), \tilde{\sigma}) + v(u)$ for every $u \in AE$ and $x \in \tilde{V}$, we see that ℓ_1, ℓ_2 and v are smooth functions. It follows from (3) of Proposition 2.4 and the identity $D_u \hat{\alpha} = d_u \hat{\alpha} + \omega \cdot \hat{\alpha}(u)$ that

$$\begin{aligned}D_u \hat{\alpha} =& d_u \hat{\alpha} + [\omega_L, \ell_2(u)] \circ \tilde{\pi}_2 - [\omega_L, \ell_1(u)] \circ \tilde{\pi}_1 \\ & - [\ell_2(u) \circ \omega_T - \ell_1(u) \circ \omega_T] + \omega_L \cdot v(u)\end{aligned} \tag{4.3}$$

where $\omega = \omega_L + \omega_T$.

Since our primary objective is to develop an appropriate arena for affine geometry we are mostly interested in maps $\hat{\alpha}$ which have their values in $\mathcal{D}_s(\tilde{E}, \tilde{V})$, the set of difference functions on $\tilde{E}$. There is no appreciable reduction of (4.2) for this case, but if $\hat{\alpha}$ has its values in the set $\mathrm{Aff}^2(\tilde{E}, \tilde{V}) \cap \mathcal{D}_s(\tilde{E}, \tilde{V})$ of affine difference functions then (4.3) reduces further and we have the result:

PROPOSITION 4.1. *Assume that* $\hat{\alpha} : AE \to \mathcal{V}_s^2(\tilde{E}, \tilde{V})$ *is an equivariant mapping with values in the set* $\mathrm{Aff}^2(\tilde{E}, \tilde{V}) \cap \mathcal{D}_s(\tilde{E}, \tilde{V})$ *of affine difference functions. Then*

(1) *there is a function* ℓ *from* AE *into* $G\ell(\tilde{V})$ *such that*
   (i) $\ell(ua) = g^{-1}\ell(u)g$, *and*
   (ii) $\hat{\alpha}(u) = \ell(u)\tilde{\delta}$ *for every* $u \in AE$ *and* $a = (v, g) \in A_n$,
(2) $D_u\hat{\alpha} = d_u\hat{\alpha} + [\omega_L, \ell(u)]\tilde{\delta}$, *and*
(3) $D_u\hat{\alpha} = (D_u\ell)\ell(u)^{-1} \cdot \hat{\alpha}(u)$ *for all* $u \in AE$.

Proof. To see that (1) (ii) is true observe that if $\hat{\alpha}$ is affine we know that

$$\hat{\alpha}(u) = \ell_2(u) \circ \tilde{\delta}_{\bar{\sigma}} \circ \pi_2 - \ell_1(u) \circ \tilde{\delta}_{\bar{\sigma}} \circ \pi_1 + v(u)$$

for linear maps $\ell_1(u)$, $\ell_2(u)$ and some vector $v(u)$. Since $\hat{\alpha}(u)$ is assumed to be a difference function it follows that $v(u) = \hat{\alpha}(u)(\bar{\sigma}, \bar{\sigma}) = 0$. Moreover, we also have that $\hat{\alpha}(u)(x, \bar{\sigma}) = -\hat{\alpha}(u)(\bar{\sigma}, x)$ for all $x \in \tilde{E}$, from which it follows that $\ell_1(u)(\tilde{\delta}_{\bar{\sigma}}(x)) = \ell_2(u)(\tilde{\delta}_{\bar{\sigma}}(x))$ for all $x \in \tilde{E}$. Thus $\ell_1 = \ell_2$ and

$$\hat{\alpha}(u) = \ell_1(u) \circ (\tilde{\delta}_{\bar{\sigma}} \circ \pi_2 - \tilde{\delta}_{\bar{\sigma}} \circ \pi_1) = \ell_1(u)\tilde{\delta}.$$

To see that (1)(i) is true observe that the equivariance of $\hat{\alpha}$ implies that $\ell(ua) \circ \tilde{\delta} = a^{-1} \cdot [\ell(u) \circ \tilde{\delta}]$ for all $u \in AE$ and $a = (v, g) \in A_n$. Now

$$\begin{aligned}(a^{-1} \cdot [\ell(u) \circ \tilde{\delta}])(\tilde{\eta}, \tilde{\xi}) &= g^{-1}\ell(u)(\tilde{\delta}_{\bar{\sigma}}(a\tilde{\xi}) - \tilde{\delta}_{\bar{\sigma}}(a\tilde{\eta})) \\ &= g^{-1}\ell(u)(g\tilde{\delta}_{\bar{\sigma}}(\tilde{\xi}) - g\tilde{\delta}_{\bar{\sigma}}(\tilde{\eta})) \\ &= (g^{-1}\ell(u)g)\tilde{\delta}(\tilde{\eta}, \tilde{\xi}).\end{aligned}$$

It follows that $\ell(ua) = g^{-1}\ell(u)g$ and (1) is proven. Statement (2) follows easily from formula (4.3) and the fact that $\ell_1 = \ell_2$. Finally, (3) follows from the equation

$$D\hat{\alpha} = d\hat{\alpha} + [\omega_L, \ell]\tilde{\delta} = (d\ell + [\omega_L, \ell])\tilde{\delta} = ((D\ell)\ell^{-1})\hat{\alpha}.$$

Thus Proposition 4.1 is established. ∎

At this point we consider the problem of how one covariantly differentiates a section α of $\mathcal{V}^2(E,V)$ with respect to a vector field X defined on M. We also derive a formula for this derivative which is analogous to a corresponding formula for $\nabla_X g$ which arises in metric geometry.

In particular we recall that if g is a metric and X and Y are vector fields then

$$(\nabla_Z g)(X,Y) = \nabla_Z(g(X,Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$$

for each vector field Z. Metric geometries are characterized by requiring that $\nabla_Z g = 0$ for all Z. It is our purpose, in the next few paragraphs, to develop a formula for difference function fields analogous to this known formula for metrics. An equation of the type $\nabla_X \alpha = 0$ for all X would then define «affine geometries» at a more primitive level than may be accomplished using metrics.

The formula we derive states that for every vector field Z and sections μ and ν of $(\overline{E},\overline{V},\overline{\delta},\overline{\sigma}) \cong (E,V,\delta,\sigma)$,

$$(\nabla_Z \alpha)(\mu,\nu) = \nabla_Z(\alpha(\mu,\nu)) - d\alpha(\nabla_X \mu, \nabla_X \nu) .$$

To make sense of this formula we need to know how to differentiate sections of the vector bundle $\mathcal{V}^2(E,V)$.

Recall from [10] (page 115) how the covariant derivative of a section of a vector bundle is related to the exterior covariant derivative of its corresponding equivariant mapping. If (P,M,π) is a principal bundle, $G \times \tilde{V} \to \tilde{V}$ is an action of the structure group G of P on a vector space $\tilde{V}$, and V is the corresponding vector bundle, then covariant derivatives of sections σ of V correspond to covariant derivatives of equivariant mappings $\psi_\sigma : P \to \tilde{V}$ via the formula

$$(\nabla_X \sigma)_p = [u, X_u^* \psi_\sigma] = [u, D_u \psi_\sigma(X_u^*)]$$

for $u \in P$, $p = \pi(u)$, $X \in T_{\pi(u)}M$ and X_u^* the horizontal lift of X to $u \in P$.

DEFINITION 4.2. Let ω be any connection on AE. If α is a section of the vector bundle $\mathcal{V}^2(E,V)$ and X is a tangent vector to M at $p \in M$ we define $\nabla_X \alpha$ by the formula $(\nabla_X \alpha)_p = [u, (D_u \hat{\alpha})(X_u^*)]$ where $u \in AE$ such that $\pi(u) = p$, $\hat{\alpha}$ is the equivariant mapping from AE into $\mathcal{V}_s^2(\tilde{E},\tilde{V})$ corresponding to α, and X_u^* is the ω-horizontal lift of X to u.

If X is a vector field on M and α is a section of $\mathcal{V}^2(E,V)$, then $\nabla_X \alpha$ is a section of $AE \times_{A_n} \mathcal{V}_s^2(\tilde{E},\tilde{V})$ which we have identified with $\mathcal{V}^2(E,V) \cong \mathcal{V}^2(\overline{E},\overline{V})$. Relative to this identification we have the formula

$$\begin{aligned}(\nabla_X \alpha)([u,\tilde{\eta}],\ [u,\tilde{\xi}]) &= (u, (D_u \hat{\alpha})(X_u^*))^\star([u,\tilde{\eta}],[u,\tilde{\xi}]) \\ &= [u, (D_u \hat{\alpha})(X_u^*)(\tilde{\eta},\tilde{\xi})]\end{aligned} \tag{4.4}$$

for $u \in AE$, $\tilde{\eta},\tilde{\xi} \in \tilde{E}$. Here X^* is the horizontal lift of the vector field X to AE and the symbol $\star$ is defined by equation (3.10).

THEOREM 4.2. *If* μ *and* ν *are sections of the affine bundle* (E, V, δ, σ) *and* α *is a section of* $\mathcal{V}^2(E, V)$ *, then*

$$(\nabla_X \alpha)(\mu, \nu) = \nabla_X(\alpha(\mu, \nu)) - d\alpha(\nabla_X \mu, \nabla_X \nu)$$

for every vector field X *on* M .

REMARK. Before we proceed with the proof of Theorem 4.1 we clarify what we mean by $d\alpha(\nabla_X \mu, \nabla_X \nu)$. The function $d\alpha(\nabla_X \mu, \nabla_X \nu)$ is a compact notation for the section of V defined by

$$d\alpha(\nabla_X \mu, \nabla_X \nu)(p) = d_{(\mu(p), \nu(p))}(\alpha(p))((\nabla_X \mu)(p), (\nabla_X \nu)(p))$$

for all $p \in M$.

We find the following lemma useful in the proof of the Theorem.

LEMMA 4.3. *If* $u \in AE$, $p = \pi(u)$ *, and* X_u^* *is the horizontal lift of* X_p *to* u *, then*

$$d\alpha(\nabla_X \mu, \nabla_X \nu)(p) =$$
$$[u, d_{(\hat{\mu}(u), \hat{\nu}(u))}(\hat{\alpha}(u))(D_u \hat{\mu}(X_u^*), D_u \hat{\nu}(X_u^*))]$$

where $\hat{\mu}, \hat{\nu}, \hat{\alpha}$ *are the equivariant mappings corresponding to the sections* μ, ν, α *, respectively.*

Proof. Let $(\tilde{\eta}_o, \tilde{\xi}_o) \in \tilde{E} \times \tilde{E}$, $v_o = [u, \tilde{\eta}_o]$, $\omega_o = [u, \tilde{\xi}_o]$. We first show that

$$d_{(v_o, w_o)}(\alpha(p))([u, \tilde{\eta}], [u, \tilde{\xi}]) = [u, d_{(\tilde{\eta}_o, \tilde{\xi}_o)}(\hat{\alpha}(u))(\tilde{\eta}, \tilde{\xi})]$$

for all $\tilde{\eta}, \tilde{\xi} \in \tilde{E}$ (here we identify (E, V, δ, σ) with $(\overline{E}, \overline{V}, \overline{\delta}, \overline{\sigma})$). Let $\lambda, \kappa : (-\epsilon, \epsilon) \to \tilde{E}$ be curves in $\tilde{E}$ such that $\lambda(0) = \tilde{\eta}_o$, $\kappa(0) = \tilde{\xi}_o$, and $\frac{d\lambda}{dt}(0) = \tilde{\eta}$, $\frac{d\kappa}{dt}(0) = \tilde{\xi}$ (here we identify $T_{\eta_o}\tilde{E}$ and $T_{\xi_o}\tilde{E}$ with $\tilde{E}$ via the mappings $\tilde{\delta}_{\tilde{\sigma}}^{-1} \circ d_{\tilde{\eta}_0} \tilde{\delta}_{\tilde{\sigma}}$ and $\tilde{\delta}_{\tilde{\sigma}}^{-1} \circ d_{\tilde{\xi}_o} \tilde{\delta}_{\tilde{\sigma}}$ respectively). We have that

$$\alpha(p)([u, \lambda(t)], [u, \kappa(t)]) = [u, \hat{\alpha}(u)(\lambda(t), \kappa(t))]$$

for $t \in (-\epsilon, \epsilon)$. Differentiating at $t = 0$ yields

$$d_{(v_o, w_o)}(\alpha(p))([u, \tilde{\eta}], [u, \tilde{\xi}]) = [u, d_{(\tilde{\eta}_o, \tilde{\xi}_o)}(\hat{\alpha}(u))(\tilde{\eta}, \tilde{\xi})]$$

Since $\mu(p) = [u, \hat{\mu}(u)]$, $\nu(p) = [u, \hat{\nu}(u)]$ for $u \in AE$, $p = \pi(u)$ this result implies that

$$\begin{aligned} &d(\alpha(p))((\nabla_X \mu)(p), (\nabla_X \nu)(p)) \\ &= d_{(\mu(p),\nu(p))}(\alpha(p))([u, D_u\hat{\mu}(X_u^*)], [u, D_u^\nu(X_u^*)]) \\ &= [u, d_{(\hat{\mu}(u),\hat{\nu}(u))} d(\hat{\alpha}(u))(D_u\hat{\mu}(X_u^*), D_u\hat{\nu}(X_u^*))] \,. \end{aligned}$$

The lemma follows.

Proof of Theorem 4.2. In order to establish the result $\nabla_X \alpha(\mu, \nu) = \nabla_X(\alpha(\mu, \nu)) - d\alpha(\nabla_X \mu, \nabla_X \nu)$ it is convenient to write each term in terms of the exterior covariant derivative on the bundle. The lemma does this for $d\alpha(\nabla_X \mu, \nabla_X \nu)$. It was shown in Proposition 3.6 that $\hat{\alpha}(\hat{\mu}, \hat{\nu})$ is the equivariant mapping corresponding to the section $\alpha(\mu, \nu)$ of $V = \overline{V}$. Thus, by definition,

$$\nabla_X(\alpha(\mu, \nu))(p) = [u, D_u(\hat{\alpha}(\hat{\mu}, \hat{\nu}))(X_u^*)] \,.$$

It follows from Equation (4.4) that

$$(\nabla_X \alpha)_p(\mu(p), \nu(p)) = [u, (D_u\hat{\alpha})(X_u^*)(\hat{\mu}(u), \hat{\nu}(u))] \,.$$

Thus the identity we are trying to establish reduces to

$$\begin{aligned} &(D_u\hat{\alpha})(X_u^*)(\hat{\mu}(u), \hat{\nu}(u)) = D_u(\hat{\alpha}(\hat{\mu}, \hat{\nu}))(X_u^*) \\ &- d_{(\hat{\mu}(u),\hat{\nu}(u))}(\hat{\alpha}(u))((D_u\hat{\mu})(X_u^*), (D_u\hat{\nu}(X_u^*)) \,. \end{aligned}$$

Since X_u^* is horizontal the latter identity is equivalent to

$$\begin{aligned} &(d_u\hat{\alpha})(X_u^*)(\hat{\mu}(u), \hat{\nu}(u)) = d_u(\hat{\alpha}(\hat{\mu}, \hat{\nu}))(X_u^*) \\ &- d_{(\hat{\mu}(u),\hat{\nu}(u))}(\hat{\alpha}(u))(d_u\hat{\mu}(X_u^*), d_u\hat{\nu}(X_u^*)) \,. \end{aligned}$$

It remains only to verify this identity. Let $f : AE \times AE \to \tilde{V}$ be defined by

$$f(v, w) = \hat{\alpha}(v)(\hat{\mu}(w), \hat{\nu}(w))$$

and let $g : AE \to \tilde{V}$ be given by $g(u) = f(u, u) = \hat{\alpha}(u)(\hat{\mu}(u), \hat{\nu}(u))$. Now $d_u g = (d_1 f)_{(u,u)} + (d_2 f)_{(u,u)}$ where $(d_1 f)_{(v,w)}$ is the exterior derivative of the map $x \to f(x, w)$ at $x = v$ and $(d_2 f)_{(v,w)}$ is the exterior derivative of the map $y \to f(v, y)$ at $y = w$. But $f(x, w) = \hat{\alpha}(x)(\hat{\mu}(w), \hat{\nu}(w)) = (ev_{(\hat{\mu}(w),\hat{\nu}(w))} \circ \hat{\alpha})(x)$ implies that $(d_1 f)_{(v,w)}(X) = d_v[ev_{(\hat{\mu}(w),\hat{\nu}(w))} \circ \hat{\alpha}](X) = [ev_{(\hat{\mu}(w),\hat{\nu}(w))} \circ d_v\hat{\alpha}](X) =$

$(d_v\hat{\alpha})(X)(\hat{\mu}(w),\hat{\nu}(w))$ for every $(v,w) \in AE \times AE$ and $X \in T_vAE$. Thus we have

$$(d_1 f)_{(u,u)}(X) = (d_u\hat{\alpha})(X)(\hat{\mu}(u),\hat{\nu}(u)) \tag{4.5}$$

On the other hand observe that $f(v,y) = \hat{\alpha}(v)(\hat{\mu}(y),\hat{\nu}(y)) = [\hat{\alpha}(v) \circ (\hat{\mu} \times \hat{\nu})](y)$ for each $y \in AE$, and consequently

$$\begin{aligned}(d_2 f)_{(v,w)}(Y) &= d_{(\hat{\mu}(w),\hat{\nu}(w))}(\hat{\alpha}(v)) \circ d_w(\hat{\mu} \times \hat{\nu})](Y) \\ &= [d_{(\hat{\mu}(w),\hat{\nu}(w))}(\hat{\alpha}(v)) \circ (d_w\hat{\mu} \times d_w\hat{\nu})](Y)\end{aligned}$$

for all $v, w \in AE$, $Y \in T_wAE$. Thus we have

$$(d_2 f)_{(u,u)}(Y) = d_{(\hat{\mu}(u),\hat{\nu}(u))}(\hat{\alpha}(u))(d_u\hat{\mu}(Y), d_u\hat{\nu}(Y)) . \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\begin{aligned}d_u g(Z) =&(d_u\hat{\alpha})(Z)(\hat{\mu}(u),\hat{\nu}(u)) \\ &+ d_{(\hat{\mu}(u),\hat{\nu}(u)}(\hat{\alpha}(u))(d_u\hat{\mu}(Z), d_u\hat{\nu}(Z))\end{aligned}$$

for each $u \in AE$, $Z \in T_uAE$. Thus

$$\begin{aligned}&d_u(\hat{\alpha}(\hat{\mu},\hat{\nu}))(Z) = (d_u\hat{\alpha})(Z)(\hat{\mu}(u),\hat{\nu}(u)) \\ &+ d_{(\hat{\mu}(u),\hat{\nu}(u))}(\hat{\alpha}(u))((d_u\hat{\mu})(Z)(d_u\hat{\nu})(Z))\end{aligned}$$

and the theorem follows. ∎

REMARK. Recall from Section 2 that if $(\tilde{\eta},\tilde{\xi}) \in \tilde{E} \times \tilde{E}$ and $u \in AE$, then $d_{(\tilde{\eta},\tilde{\xi})}(\hat{\alpha}(u))$ is regarded as a mapping from $\tilde{E} \times \tilde{E}$ to $\tilde{V}$ via its identification with $\tilde{d}_{(\tilde{\eta},\tilde{\xi})}(\hat{\alpha}(u))$ which is defined by

$$\tilde{d}_{(\tilde{\eta},\tilde{\xi})}(\hat{\alpha}(u)) \equiv d_{(\tilde{\eta},\tilde{\xi})}(\hat{\alpha}(u)) \circ [((d_{\tilde{\eta}}\tilde{\delta}_{\tilde{\sigma}})^{-1} \circ \tilde{\delta}_{\tilde{\sigma}} \times ((d_{\tilde{\xi}}\tilde{\delta}_{\tilde{\sigma}})^{-1} \circ \tilde{\delta}_{\tilde{\sigma}})].$$

Although we do not explicitly use the fact, it is interesting to note that for fixed $(\tilde{\eta},\tilde{\xi}) \in \tilde{E} \times \tilde{E}$ the mapping from AE into $\mathcal{V}_s^2(\tilde{E},\tilde{V})$ defined by

$$u \to \tilde{d}_{(\tilde{\eta},\tilde{\xi})}(\hat{\alpha}(u))$$

is an equivariant mapping relative to the action of A_n on $\mathcal{V}_s^2(\tilde{E},\tilde{V})$ defined by

$$[(v,g) \cdot \beta](\tilde{\eta}_1,\tilde{\xi}_1) = g\beta(\tilde{\delta}_{\tilde{\sigma}}^{-1}(g^{-1} \cdot \tilde{\delta}_{\tilde{\sigma}}(\tilde{\eta}_1)), \tilde{\delta}_{\tilde{\sigma}}^{-1}(g^{-1} \cdot \tilde{\delta}_{\tilde{\sigma}}(\tilde{\xi}_1)))$$

for $(v, g) \in A_n$, $\beta \in \mathcal{V}_s^2(\tilde{E}, \tilde{V})$, $(\tilde{\eta}_1, \tilde{\xi}_1) \in \tilde{E}^2$. Thus if $\tilde{\alpha}$ is equivariant relative to the «usual» action of A_n on $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$, then the map $u \to d_{(\tilde{\eta}, \tilde{\xi})}(\hat{\alpha}(u))$ is equivariant relative to a modification of the usual action which gets rid of the «translational part» of the action. The proof of this remark is easy but tedious and is left to the reader.

Recall that if g is a metric on a manifold M and $\hat{g}$ is the corresponding equivariant mapping from the frame bundle LM to $T_2^o\mathbf{R}^n$, then a connection ω is a *metric connection* iff ω reduces to the orthonormal frame bundle and this is true iff $D^\omega \hat{g} = 0$. In this case the orthonormal frame bundle is a «level surface» of the mapping $\hat{g}$.

In our present context note that if $\hat{\alpha} : AE \to \mathcal{V}_s^2(\tilde{E}, \tilde{V})$ is equivariant and if $\hat{\alpha}(AE)$ is a single orbit of $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$ relative to the action of A_n on $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$, then for each $\alpha_o \in \hat{\alpha}(AE)$, $\hat{\alpha}^{-1}(\alpha_o)$ is a subbundle of AE with structure group the isotropy subgroup of α_o (see [15], page 297). In such a case we say that $\hat{\alpha}$ *reduces* AE. In the case that $\hat{\alpha}$ reduces AM a given connection ω on AM may or may not reduce to the subbundle $\hat{\alpha}^{-1}(\alpha_o)$. In fact it is well-known that ω reduces to $\hat{\alpha}^{-1}(\alpha_o)$ iff $D^\omega \hat{\alpha} = 0$ (see [15], page 298).

One feature occurs in the present context which has no parallel in the metric case. We have a given difference function $\tilde{\delta}$ on $\tilde{E} \times \tilde{E}$ which has played a central role in all that we have done (recall that the action of A_n on $\tilde{E}$ depends on $\tilde{\delta}$ and thus so does the extended action of A_n on $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$). We may define a mapping $\hat{\delta} : AE \to \mathcal{V}_s^2(\tilde{E}, \tilde{V})$ by requiring that $\hat{\delta}(u) = \tilde{\delta}$ for all $u \in AE$. It follows from (3.4) of Section 3 that $a \cdot \tilde{\delta} = \tilde{\delta}$ for all $a \in A_n$ and consequently $\hat{\delta}$ is equivariant. Moreover, the fact that $a \cdot \tilde{\delta} = \tilde{\delta}$ for all $a \in A_n$ implies that $b \cdot \tilde{\delta} = 0$ for all $b \in a_n$. Thus, for any connection ω on AE, $D^\omega \hat{\delta} = 0$. It follows that for this special mapping $\hat{\delta}$ no real reduction actually occurs although the formal definition of bundle reduction is satisfied.

On the other hand it is easy to see that there exist difference function fields which do reduce AE. This is the case if α is a difference function field whose corresponding equivariant mapping $\hat{\alpha}$ carries AE onto an orbit of the difference function $\alpha_o = \ell_o \tilde{\delta}$ for some linear mapping ℓ_o from $\tilde{V}$ to $\tilde{V}$.

THEOREM 4.4. *Let $\ell_o : \tilde{V} \to \tilde{V}$ be any linear mapping and let $\alpha_o = \ell_o \tilde{\delta}$. An equivariant mapping $\hat{\alpha} : AE \to \mathcal{V}_s^2(\tilde{E}, \tilde{V})$ carries AE onto the orbit of $\ell_o \tilde{\delta}$ iff $\hat{\alpha}^{-1}(\ell_o \tilde{\delta})$ is a subbundle of AE with structure group*

$$G_{\ell_o} = \{(v, g) \in A_n | [g, \ell_o] = 0\}.$$

If $\hat{\alpha}$ carries AE onto the orbit of $\ell_o \hat{\delta}$ where ℓ_o is bijective and if we define $\ell : AE \to \mathcal{V}(\tilde{V})$ by $\hat{\alpha}(u) = \ell(u)\tilde{\delta}$, then ℓ is an equivariant mapping with respect to the action of A_n on $G\ell(\tilde{V})$ defined by $(v, g) \cdot h = gh^{-1}g$. Moreover, any given connection ω on AE reduces to $\hat{\alpha}^{-1}(\ell_o \tilde{\delta})$ iff $0 = D^\omega \ell = d\ell + [\omega_L, \ell]$.

Proof. First observe that if $a = (v, g) \in A_n$ and $(\tilde{\eta}, \tilde{\xi}) \in \tilde{E} \times \tilde{E}$ then $(a \cdot \alpha_o)(\tilde{\eta}, \tilde{\xi}) = (g\ell_o g^{-1})(g\tilde{\delta}(a^{-1}\tilde{\eta}, a^{-1}\tilde{\xi})) = (g\ell_o g^{-1})(a \cdot \tilde{\delta})(\tilde{\eta}, \tilde{\xi}) = (g\ell_o g^{-1})\tilde{\delta}(\tilde{\eta}, \tilde{\xi})$. Thus the orbit of α_o is given by

$$A_n \cdot \alpha_o = \{(g\ell_o g^{-1})\tilde{\delta} | g \in G\ell(\tilde{V})\}.$$

The same computation shows that the isotropy subgroup of α_o, defined to be the set of all $a = (v, g) \in A_n$ such that $a \cdot \alpha_o = \alpha_o$, is precisely the set of all (v, g) such that $g\ell_o g^{-1} = \ell_o$. Thus if $\hat{\alpha}$ is an equivariant mapping which carries AE onto the orbit of some affine difference function $\alpha_o = \ell_o\tilde{\delta}$ then $\hat{\alpha}$ has the finite dimensional vector space $g\ell(\tilde{V})\tilde{\delta}$ as its range and consequently $\hat{\alpha}$ reduces AE to a subbundle of AE having as structure group the set of all $(v, g) \in A_n$ such that $[g, \ell_o] = 0$ (see [15], page 297). If we define an action of A_n on $G\ell(\tilde{V})$ by $(v, g) \cdot h = ghg^{-1}$ for $(v, g) \in A_n$, $h \in G\ell(\tilde{V})$, then the computation in the first sentence of this proof also shows that if we define $\ell : AE \to G\ell(\tilde{V})$ by requiring that $\hat{\alpha}(u) = \ell(u)\tilde{\delta}$ then $\ell(ua) = g^{-1}\ell(u)g = a^{-1} \cdot \ell(u)$ for all $a \in A_n$, $u \in AE$. The fact that ω reduces to $\hat{\alpha}^{-1}(\ell_o\tilde{\delta}) = \ell^{-1}(\ell_o)$ iff $D^\omega \ell = 0$ is well-known (see [15]). ∎

5. AFFINE STRUCTURE MAPS AND THEIR COVARIANT DERIVATIVES

In the physical theory [13] which motivates this investigation one wishes to allow the possibility that both the difference function field δ and the section σ of an affine bundle (E, V, δ, σ) be variable. This situation is analogous to that which occurs in general relativity. In that theory the arena is a spacetime manifold in which the metric is not given directly but rather is selected by Einstein's equations along with appropriate boundary conditions. Thus one considers the class of all metrics on the given spacetime manifold and utilizes a variational principle to select the physical metric. In the affine unified theory of gravity and electromagnetism initiated in [13] it is not yet clear what the basic variables will be. It is likely that the metric g, the difference function field δ, and the section σ will all play a fundamental role. It is probable that the triple (g, δ, σ) will participate in some way in a variational procedure to select the appropriate metric-affine theory to describe gravity and electromagnetism.

Given an affine bundle (E, V, δ, σ) we observed in Section 2 that δ defines a diffeomorphism $\varphi_\delta : E \to V$ which is fiber preserving and which carries σ to the 0-section of V. This mapping is defined by $\varphi_\delta(\xi) = \delta_{\pi_E(\xi)}(\sigma(\pi_E(\xi)), \xi)$. Conversely, if $\varphi : E \to V$ is any fiber preserving diffeomorphism, then a difference field δ_φ may be defined from $E \times E$ to V by

$$(\delta_\varphi)_p(\eta, \xi) = \varphi(\xi) - \varphi(\eta).$$

At first sight it appears not to matter which formalism one uses, but we claim that the mappings $\varphi : E \to V$ which are fiber preserving have more potential information in them. Our reason is that the difference function field δ_φ which corresponds to φ ignores any changes in φ due to a translation along the fibers of V. More precisely, if $\psi : E \to V$ is defined by $\psi(\xi) = \varphi(\xi) + \lambda(\pi_E(\xi))$ where $\lambda : M \to V$ is an arbitrary section of V, then for all ξ, η in the same fiber of E we have

$$\delta_\psi(\eta,\xi) = \psi(\xi) - \psi(\eta) = \varphi(\xi) - \varphi(\eta) = \delta_\varphi(\eta,\xi)$$

so that $\delta_\psi = \delta_\varphi$. The covariant derivatives of ψ and φ will be distinguished by the covariant derivatives of their translational parts whereas the covariant derivatives of δ_ψ and δ_φ are, of course, identical. This difference shows up most clearly for the case of affine difference function fields especially when one compares Proposition 4.1 (2) to Theorem 5.2 below.

Thus in this section we derive formulas for the covariant derivatives of fiber preserving diffeomorphisms from E to V. We will also briefly discuss the question as to what functions should play the role of the «components» of a difference function field relative to the choice of an affine frame at each point of M.

DEFINITION 5.1. Assume that E is a fiber bundle and that V is a vector bundle. A smooth mapping φ from E to V is called an *affine structure mapping* iff it is a fiber-preserving diffeomorphism. In this case we refer to δ_φ as the *corresponding affine structure* on E. ∎

In the next few paragraphs we derive a formula for the covariant derivative of arbitrary affine structure mappings relative to a given connection ω on AE. The procedure is simplified if we make some identifications.

As before let $\tilde{E}$ be the standard fiber of E and $\tilde{V}$ the standard fiber of V. Let $\{r_k\}$ be a fixed basis of $\tilde{V}$ and identify $\tilde{V}$ with $\mathbf{R}^n$. We distinguish two actions of A_n on $\tilde{V} \cong \mathbf{R}^n$. The first of these actions is called the *linear action* and is defined by $(v,g) \cdot w = gw$ for $(v,g) \in A_n$, $w \in \tilde{V} \cong \mathbf{R}^n$. The second action is called the *affine action* of A_n on $\tilde{V}$ and is defined by $(v,g) \cdot w = gw + v$ for $(v,g) \in A_n$, $w \in \tilde{V}$.

If $(\tilde{E}, \tilde{V}, \tilde{\delta}, \tilde{\sigma})$ is a fixed pointed affine space, then $\tilde{E}$ may be identified with $\tilde{V}$ via the mapping $\tilde{\delta}_{\tilde{\sigma}}$. Moreover, the action of A_n on $\tilde{E}$ defined by (2.1) in Section 2 may be identified via $\tilde{\delta}_{\tilde{\sigma}}$ with the affine action defined on $\tilde{V}$ as in the last paragraph. It follows then that if we utilize the linear action of A_n on $\tilde{V}$, then $\overline{V} \equiv AE \times_{A_n} \tilde{V}$ may be identified with V as in Section 2; but if we utilize the affine action of A_n on $\tilde{V}$, then $\overline{E} = AE \times_{A_n} \tilde{E} \cong AE \times_{A_n} \tilde{V}$ may be identified with E.

Thus if $\varphi : E \to V$ is an affine structure mapping, then we may regard it as a mapping from $AE \times_{A_n} \tilde{V}$ to $AE \times_{A_n} \tilde{V}$ where the affine action is used on the domain of φ and the linear action is used on the range of φ. Clearly there is a one-to-one correspondence between affine structure mappings and equivariant mappings from AE

into Diff $(\tilde{V})$ where the action of A_n on Diff $(\tilde{V})$ is defined by

$$[(v,g)\cdot f](x) = gf(g^{-1}(x-v)) \tag{5.1}$$

for $(v,g) \in A_n$, $f \in \mathrm{Diff}(\tilde{V})$, and $x \in \tilde{V}$. This correspondence is the usual one whereby φ corresponds to $\hat{\varphi}$ iff

$$\varphi([u,x]) = [u, \hat{\varphi}(u)(x)] \tag{5.2}$$

for all $u \in AE$, $x \in \tilde{V}$.

By analogy with tensors we require that whenever $\varphi : E \to V$ is a fiber-preserving diffeomorphism and X is a vector field on M, then $\nabla_X \varphi$ is a fiber-preserving smooth mapping. Comparison with Definition 4.2 leads us to *define* $\nabla_X \varphi$ to be the fiber-preserving smooth mapping from E to V which satisfies the identity

$$(\nabla_X \varphi)([u,x]) = [u, (D_u\hat{\varphi})(X_u^*)] \tag{5.3}$$

where $u \in AE$, $x \in \tilde{V}$, and X_u^* is the horizontal lift of $X_{\pi(u)}$ to u. Here $D\hat{\varphi}$ is defined by the usual formula:

$$(D_u\hat{\varphi})(Y) = (d_u\hat{\varphi})(horY) \tag{5.4}$$

for $u \in AE$, $Y \in T_uAE$. As before hor(Y) denotes the horizontal component of Y (see [10]).

Clearly there are a number of conditions which must be met in order for the above definitions to be well-defined and for the various maps to be smooth, but one establishes these conditions by analogy with the corresponding results for $\mathcal{V}_s^2(\tilde{E},\tilde{V})$. The infinite dimensional vector space $\mathcal{M}_s(\tilde{V})$ of all smooth maps from $\tilde{V}$ to $\tilde{V}$ plays the same role in this formulation as does $\mathcal{V}_s^2(\tilde{E},\tilde{V})$ in the discussion in Section 4. The results need only minor modification and the details are left to the reader.

The problem of differentiating affine structure mappings is thus reduced to finding a formula for $D\hat{\varphi}$. We have the usual formula

$$D\hat{\varphi} = d\hat{\varphi} + \omega \cdot \hat{\varphi} \tag{5.5}$$

in the present case. Thus we need only clarify how the Lie algebra a_n of A_n acts on the space $\mathcal{M}_s(\tilde{V})$ of smooth mappings from $\tilde{V}$ to $\tilde{V}$. As usual, for $\tilde{a} = (\tilde{v},\tilde{g}) \in a_n$ choose a one-parameter group $a(t) = (v(t), g(t))$ in A_n such that $\frac{da}{dt}(0) = \tilde{a}$ and define $(\tilde{a}\cdot f)(x) = \frac{d}{dt}[a(t)\cdot f](x)|_{t=0}$ for each $f \in \mathcal{M}_s(\tilde{V})$ and $x \in \tilde{V}$. We see that

$$
\begin{aligned}
[(\tilde{v},\tilde{g})\cdot f](x) &= \frac{d}{dt}[g(t)f(g(t)^{-1}(x-v(t)))]|_{t=0}\\
&= \tilde{g}f(x) + d_x f(-\tilde{g}x-\tilde{v})\\
&= \tilde{g}f(x) - d_x f(\tilde{g}x+\tilde{v})\\
&= \tilde{g}f(x) - (d_x f)(\tilde{a}\cdot x).
\end{aligned}
$$

These remarks prove the general result:

THEOREM 5.1. *Assume that* $\varphi : E \to V$ *is any affine structure mapping and that* $\hat{\varphi} : AE \to \mathrm{Diff}(\tilde{V})$ *is its corresponding equivariant mapping. For* $u \in AE$ *and* $X \in T_u AE$, $(D_u\hat{\varphi})(X)$ *is the smooth mapping from* $\tilde{V}$ *to* $\tilde{V}$ *given by*

$$
\begin{aligned}
(D_u\hat{\varphi})(X)(x) =& (d_u\hat{\varphi})(X)(x) + \omega_L(X)\hat{\varphi}(u)(x)\\
&- d_x(\hat{\varphi}(u))(\omega(X)\cdot x)
\end{aligned}
$$

for all $x \in \tilde{V}$. ■

An interesting subset of the set of all affine structure maps from a fiber bundle E into a vector bundle V is that set of maps $\varphi : E \to V$ such that $\varphi|E_p$ is an affine mapping for each $p \in M$. Observe that φ belongs to this class of maps iff its corresponding equivariant mapping $\hat{\varphi} : AE \to \mathrm{Diff}(\tilde{V})$ actually carries AE into the set $\mathrm{Aff}(\tilde{V})$ of all affine mappings from $\tilde{V}$ to $\tilde{V}$. Since $\mathrm{Aff}(\tilde{V})$ is a finite dimensional Lie group one expects such structure functions to be more tractable. Indeed if $\hat{\varphi} : AE \to \mathrm{Aff}(\tilde{V})$ is equivariant, then we may define maps $\hat{\varphi}_L : AE \to G\ell(\tilde{V})$ and $\hat{\varphi}_T : AE \to \tilde{V}$ by first defining $\hat{\varphi}_T(u) \equiv \hat{\varphi}(u)(\tilde{o})$ where $\tilde{o}$ is the zero vector in $\tilde{V}$ and then defining $\hat{\varphi}_L$ by $\hat{\varphi}_L = \hat{\varphi} - \hat{\varphi}_T$. Thus $\hat{\varphi} = \hat{\varphi}_L + \hat{\varphi}_T$ may be decomposed into a «linear» and «translational» part.

If ω is a connection on AE then ω has its values in $a_n = g\ell(n,\mathbf{R}) \oplus \mathbf{R}^n$ and consequently we have that $\omega_u(X) = \omega_L(X) + \omega_T(X)$ where $u \in AE$, $X \in T_u AE$ and $\omega_L(X) \in g\ell(n,\mathbf{R})$, $\omega_T(X) \in \mathbf{R}^n$. Since we have identified $\tilde{V}$ with $\mathbf{R}^n$ these definitions give us a way of decomposing the covariant derivative of any equivariant mapping from AE into $\mathrm{Aff}(\tilde{V})$.

THEOREM 5.2. *If* $\hat{\varphi} : AE \to \mathrm{Aff}(\tilde{V})$ *is equivariant, then* $D\hat{\varphi} = D\hat{\varphi}_L + D\hat{\varphi}_T$ *where*

$$D\hat{\varphi}_L \equiv d\hat{\varphi}_L + [\omega_L, \varphi_L] \tag{5.6a}$$

$$D\hat{\varphi}_T = d\hat{\varphi}_T + \omega_L\hat{\varphi}_T - \hat{\varphi}_T\omega_T\,. \tag{5.6b}$$

Proof. It follows from Theorem 5.1 that

$$D_u\hat{\varphi}(X)(x) =(d_u\hat{\varphi})(X)(x) + \omega_L(X)\hat{\varphi}(u)(x)$$
$$- d_x(\hat{\varphi}(u))(\omega(X)\cdot x)$$

for $u \in AE$, $X \in T_uAE$, $x \in \tilde{V}$. Now $\hat{\varphi}(u) = \hat{\varphi}_L(u) + \hat{\varphi}(u)(\tilde{o})$ and consequently $d_x(\hat{\varphi}(u)) = d_x(\hat{\varphi}_L(u)) = \hat{\varphi}_L(u)$ since $\hat{\varphi}_L(u)$ is a *linear* map from $\tilde{V}$ to $\tilde{V}$. Thus

$$\begin{aligned}(D_u\hat{\varphi})(X) &=(d_u\hat{\varphi})(X) + \omega_L(X)\hat{\varphi}(u)\\ &\quad - \hat{\varphi}_L(u)(\omega_L(X) + \omega_T(X))\\ &=(d_u\hat{\varphi}_L)(X) + (d_u\hat{\varphi}_T)(X)\\ &\quad + \omega_L(X)\hat{\varphi}_L(u) + \omega_L(X)\hat{\varphi}_T(u)\\ &\quad - \hat{\varphi}_L(u)(\omega_L(X)) - \hat{\varphi}_L(u)\omega_T(X)\\ &=(d_u\hat{\varphi}_L)(X) + [\omega_L(X), \hat{\varphi}_L(u)]\\ &\quad + (d_u\hat{\varphi}_T)(X) + \omega_L(X)\hat{\varphi}_T(u)\\ &\quad - \hat{\varphi}_L(u)\omega_T(X)\ .\end{aligned}$$

To conclude the proof of the theorem we must show that $D\hat{\varphi}_L = d\hat{\varphi}_L + [\omega_L, \hat{\varphi}_L]$ and that $D\hat{\varphi}_T = d\hat{\varphi}_T + \omega_L\hat{\varphi}_T - \hat{\varphi}_T\omega_T$. To establish these formula we first determine how $\hat{\varphi}_L$ and $\hat{\varphi}_T$ transform under change of frame. We have that

$$\hat{\varphi}(ua)(x) = \hat{\varphi}_L(ua)(x) + \hat{\varphi}(ua)(\tilde{o})$$

and

$$\begin{aligned}\hat{\varphi}(ua)(x) &= [a^{-1}\cdot\hat{\varphi}(u)](x)\\ &= g^{-1}\hat{\varphi}_L(u)(gx + v) + g^{-1}\hat{\varphi}(u)(\tilde{o})\\ &= g^{-1}\hat{\varphi}_L(u)(gx) + g^{-1}\hat{\varphi}_L(u)(v) + g^{-1}\hat{\varphi}(u)(\tilde{o})\end{aligned}$$

for $u \in AE$, $a = (v, g) \in A_n$, $x \in \tilde{V}$. Thus

(5.7a) $$\hat{\varphi}_L(ua)(x) = g^{-1}\hat{\varphi}_L(u)(gx)$$

(5.7b) $$\hat{\varphi}_T(ua) = g^{-1}\hat{\varphi}_T(u) + g^{-1}\hat{\varphi}_L(u)(v)$$

for $u \in AE$, $a = (v,g) \in A_n$, $x \in \tilde{V}$. Now clearly it follows from (5.7a) that $D\hat{\varphi}_L = d\hat{\varphi}_L + [\omega_L, \varphi_L]$. Because of the coupling of $\hat{\varphi}_T$ and $\hat{\varphi}_L$ in (5.7b) the formula (5.6b) is not so obvious a consequence of (5.7). To obtain (5.6b) first observe that

$$\begin{aligned}(D_u\hat{\varphi})(X)(\tilde{o}) =&(d_u\hat{\varphi})(X)(\tilde{o}) + \omega_L(X)(\hat{\varphi}(u)(\tilde{o}))\\ &- \hat{\varphi}_L(u)(\omega(X)(\tilde{o}))\end{aligned}$$

is a consequence of Theorem 5.1 and the fact that $d_x(\hat{\varphi}(u)) = \hat{\varphi}_L(u)$. But $\omega(X)(\tilde{o}) = \omega_T(X)$ and $\hat{\varphi}(u)(\tilde{o}) = \hat{\varphi}_T(u)$ imply

$$\begin{aligned}(D_u\hat{\varphi})(X)(\tilde{o}) =&ev_{\tilde{o}}((d_u\hat{\varphi})(X)) + \omega_L(X)(\hat{\varphi}_T(u))\\ &- \hat{\varphi}_L(u)(\omega_T(X))\\ =&d_u(ev_{\tilde{o}} \circ \hat{\varphi})(X) + \omega_L(X)(\hat{\varphi}_T(u))\\ &- \hat{\varphi}_L(u)(\omega_T(X))\\ =&(d_u\hat{\varphi}_T)(X) + \omega_L(X)(\hat{\varphi}_T(u))\\ &- \hat{\varphi}_L(u)(\omega_T(X)) .\end{aligned}$$

On the other hand

$$\begin{aligned}(D_u\hat{\varphi})(X)(\tilde{o}) &= (d_u\hat{\varphi})(horX)(\tilde{o})\\ &= ev_{\tilde{o}}(d_u\hat{\varphi}(horX))\\ &= d_u(ev_{\tilde{o}} \circ \hat{\varphi})(horX)\\ &= (D_u\hat{\varphi}_T)(X) .\end{aligned}$$

Thus $(D_u\hat{\varphi}_T)(X) = (D_u\hat{\varphi})(X)(\tilde{o}) = (d_u\hat{\varphi}_T)(X) + \omega_L(X)\hat{\varphi}_T(u) - \hat{\varphi}_L(u)\omega_T(X)$ and the theorem follows. ∎

If δ is a difference function field on $E \times E$ with values in V, then $\hat{\delta}$ is an equivariant mapping from AE into $\mathcal{V}_s^2(\tilde{E}, \tilde{V})$. It follows that there is an equivariant mapping $\hat{\varphi}$ from AE into $\mathcal{M}_s(\tilde{E}, \tilde{V})$ such that

$$\hat{\delta}(u)(\eta,\xi) = \hat{\varphi}(u)(\xi) - \hat{\varphi}(u)(\eta)$$

for all $u \in AE$, $\eta, \xi \in \tilde{E}$. If, in fact, we identify $\tilde{E}$ with $\tilde{V}$ as we have been doing in this section, then $\hat{\varphi}$ maps AE into $\mathrm{Diff}(\tilde{V})$. The mapping $\hat{\varphi}$ may or may not carry AE into $\mathrm{Aff}(\tilde{V})$, but if it does then we see that

$$\hat{\delta}(u)(\eta,\xi) = \hat{\varphi}_L(u)(\xi) - \hat{\varphi}_L(u)(\eta)$$

for all $u \in AE$, $\eta, \xi \in \tilde{E} \cong \tilde{V}$. Thus $\hat{\delta}$ never picks up the translational features of $\hat{\varphi}$ and, of course, this shows up in the dynamics of both $\hat{\delta}$ and δ. In fact, it follows immediately that

$$(D_u\hat{\delta})(X)(\eta,\xi) = (D_u\hat{\varphi}_L)(X)(\xi) - (D_u\hat{\varphi}_L)(\eta)$$

and since $D\hat{\varphi}_L = d\hat{\varphi}_L + [\omega_L, \varphi_L]$ depends only on the linear part of the connection ω we see that the covariant derivative of $\hat{\delta}$ essentially ignores the fact that we are working on the affine frame bundle rather than the linear frame bundle. We find that

$$D^{\omega_o}\hat{\delta} = (D^{\omega_o}\varphi_L) \circ \pi_2 - (D^{\omega_o}\varphi_L) \circ \pi_1$$

where we use ω_o instead of ω_L and where ω_o is the unique connection which agrees with ω on $LM \subseteq AE$ but which reduces to LM. The point of these remarks is that it is not clear at this point whether the appropriate arena for a unified theory of gravity and electromagnetism should be the space of triples (g, δ, σ) or the space of triples (g, φ, σ). If the theory demands that translational features play an important role then perhaps the latter space would be more appropriate.

Finally, we wish to discuss briefly the «components» of a difference function field δ. Recall that if t is a tensor field of type $\binom{r}{s}$ on a manifold M, then the corresponding equivariant mapping $\hat{t} : LM \to T^r_s\mathbf{R}^n$ picks out the components of t. In fact for a given frame $u \in LM$, $u = (p, e_i)$, one expresses $t = t^{i_1 \cdots i_r}_{j_1 \cdots j_s}(e_{i_1} \otimes \cdots e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s})$ and then $\hat{t}(u) = t^{i_1 \cdots i_r}_{j_1 \cdots j_s}(r_{i_1} \otimes \cdots \otimes r_{i_r} \otimes r^{j_1} \otimes \cdots \otimes r^{j_s})$. Thus $\hat{t}(u)$ selects the components of t in the frame u. In the present theory a difference function field is a section of an infinite dimensional bundle and thus, strictly speaking, has no components. On the other hand we regard $\hat{\delta}(u)$ as representing the «components» of δ in the affine frame u. In the case when $\hat{\delta}(u) = \hat{\varphi}(u) \circ \pi_2 - \hat{\varphi}(u) \circ \pi_1$ for some equivariant mapping $\hat{\varphi}$ from AE into $\mathrm{Aff}(\tilde{V})$, the difference function field δ becomes linear and the components of δ become meaningful. If $\{r_k\}$ is the standard basis of $\tilde{V}$, then $\hat{\varphi}_L(u)(r_k) = \varphi^\ell_k(u) r_\ell$ and we regard $\{\varphi^\ell_k(u)\}$ as being the components of the affine structure mapping φ in the frame u. By analogy with the case of tensors the components of δ in the frame u are given by $\hat{\delta}(u) = \hat{\varphi}_L(u) \circ \pi_2 - \hat{\varphi}_L(u) \circ \pi_1 = \hat{\varphi}_L(u)\delta_{\tilde{V}}$ where $\delta_{\tilde{V}}$ is the standard difference function on the vector space $\tilde{V}$, $\delta_{\tilde{V}}(x, y) = y - x$ for $x, y \in \tilde{V}$. If E^i_j is the matrix with 1 in the i-th row, j-th column and 0 elsewhere

$$\hat{\delta}(u) = \varphi^i_j(u)(E^j_i \otimes \delta_{\tilde{V}}) .$$

We regard $\{E^j_i \otimes \delta_{\tilde{V}}\}$ as a basis and $\{\varphi^i_j(u)\}$ as the components of δ in the frame u relative to this basis. The components $\{\varphi^i_j(u)\}$ transform like a tensor of type $\binom{1}{1}$ for if

$\overline{u} = ua$ is the frame gotten by transforming u by $a = (v, g)$, then $\hat{\varphi}_L(\overline{u}) = a^{-1} \cdot \hat{\varphi}_L(u)$ and

$$\hat{\varphi}_L(\overline{u}) = g^{-1} \hat{\varphi}_L(u) g$$

or

$$\varphi^i_j(\overline{u}) = g^{-1} \hat{\varphi}_L(u) g$$

or

$$\varphi^i_j(\overline{u}) = (g^{-1})^i_k \varphi^k_\ell(u) g^\ell_j.$$

Moreover, $D\hat{\delta} = (D\hat{\varphi}_L)\delta_{\bar{V}}$ and by Theorem 5.2 $D\hat{\delta} = (d\hat{\varphi}_L + [\omega_L, \hat{\varphi}_L])\delta_{\bar{V}}$. Thus if s is a local gauge in AE and $\overline{\varphi}^i_j(p) = \varphi^i_j(s(p))$, then

$$\nabla_\mu \overline{\varphi}^i_j = \partial_\mu \overline{\varphi}^i_j + \Lambda^i_{\mu k} \overline{\varphi}^k_j - \Lambda^\ell_{\mu j} \overline{\varphi}^i_\ell$$

where $\Lambda^j_{\mu i} E^i_j = \omega_L(d_p s(\partial_\mu))$.

6. REMARKS ABOUT AFFINE GEOMETRY IN PHYSICS

In this section we briefly discuss some examples which illustrate more specifically how our formalism relates to that used in certain already-developed applications of affine geometry.

We take the view that in most physical theories the concept of energy-momentum may be taken to be a primitive concept, in the sense that it is not built up from more basic concepts yet practically every physical theory requires its consideration. Moreover, in those theories in which charged particles play a role it appears that the concept of energy-momentum should be an affine rather than vector concept. In physical terms this means that observers in such theories must distinguish between the energy-momenta of charged particles and of uncharged particles and that one way of accounting for the differences is to provide for a shift of origin in energy-momentum «space». This idea was expressed explicitly for the first time in the work of Norris and his collaborators (see [9] and [13]).

From this point of view one postulates the existence of an affine bundle Π over a given space-time manifold M with the property that for $p \in M$ the fiber Π_p of Π over p represents all possible energy-momentum configurations at p. This is fully analogous to the fact that in Lagrangian mechanics the tangent bundle of a configuration space Q is a vector bundle over Q whose fiber at $q \in Q$ represents all possible generalized velocity configurations at q. Unlike the Lagrangian mechanics case, however, it is our contention that in the case of charged particle dynamics in M one should choose a formulation which permits one to consider differing choices of zero energy-momenta. It is argued in [9] that the presence of an electromagnetic field produces a shift in the energy-momenta of charged particles which then redefines what it means for a charged particle to have zero energy-momentum. Moreover, it is argued in [13] that in an appropriate «affine gauge» the electromagnetic field itself may be viewed as arising from a field which locally defines a choice of zero energy-momentum. Thus we propose that in any context where charged particle dynamics is important the notion of observer itself should incorporate the following data:

(1) a world-line in space-time,

(2) a reference frame «moving» smoothly along the world line, and

(3) a choice of zero energy-momentum at each point of the world line.

In other words we postulate that the notion of an observer is characterized by defining an ordered pair (γ, θ) where γ is a curve in the linear frame bundle LM of M and θ is a section of that part of the energy-momentum affine bundle Π which lies over the world-line $\pi \circ \gamma$ of the observer (π is the projection of LM onto M).

It should be noted that the formalism can be utilized in many situations depending on which vector bundle V serves as the model on which Π is defined. One could formulate models where V could be TM, T^*M, a spinor bundle over M, or perhaps some relatively complicated spliced bundle depending on the specific physical theory being investigated. In some of these cases it will be meaningful to consider energy-momentum configurations of sections of the relevant vector bundle.

Given a zero of energy-momentum θ of $\Pi|_{\pi\circ\gamma}$ it admits an extension to Π (recall that $\pi \circ \gamma$ defines a submanifold of M since it is a time-like world-line) and if we call such an extension θ, then the pointed bundle (Π, θ) can be identified via Proposition 3.3 with an associated bundle E of the affine frame bundle $A\Pi$ of Π. Under this identification θ is identified with the zero section of E. If $\tilde{\theta}$ is a different zero of energy-momentum of $\Pi|_{\pi\circ\gamma}$ then $(\Pi, \tilde{\theta})$ is identified with another associated bundle $\tilde{E}$ of $A\Pi$. Each of these gives a valid description of the zero of the energy-momentum of an observer and they are related via translational gauge freedom in $A\Pi$. For example, in the special case that $V = TM$ the observer (γ, θ) defines a curve in the subbundle

$$L_\theta M = \{(p, e_i, \theta(p)) \mid (p, e_i) \in LM\}$$

of AM while $(\tilde{\gamma}, \tilde{\theta})$ defines a curve in

$$L_{\tilde{\theta}} M = \{(p, e_i, \tilde{\theta}(p)) \mid (p, e_i) \in LM\}.$$

Obviously these lie in two different copies of LM inside the affine frame bundle AM of M, but there is a well-defined gauge transformation which takes either to the other in AM.

It is not clear how far this formalism should be pushed. One could take the point of view that in any theory in which energy-momentum is an affine concept, the Hamiltonian of a physical system should be defined on a momentum space whose elements are affine quantities rather than vector quantities. If so, then the Hamiltonian $\mathcal{H}$ would be a function from the momentum bundle Π into $\mathbf{R}$ where Π is not generally a vector bundle but rather is an affine bundle modeled on the vector bundle T^*M. As an example of how this would work consider first a free (uncharged) particle moving in flat Minkowski spacetime M_0. Let $\mathcal{H}_0$ be the Hamiltonian of this particle. Then $\mathcal{H}_0(q,\tilde{p}) = \sum_{i=1}^{n} \delta^i(\tilde{p}, \theta(q))$ where $q \in M_0$, $\tilde{p} \in \Pi_q$, δ is the difference function field on Π and θ represents the zero energy-momentum of the free particle. If we identify (Π, θ) with the bundle associated to AM_0 as in the last paragraph, we see that (Π, θ) may be identified with $(T^*M_0, 0)$ where 0 is the zero section of T^*M_0. Moreover, δ may be identified with the trivial difference function $(\delta_0)_q(p_1, p_2) = p_1 - p_2$. Assume now that A is a globally defined 4-potential of some electromagnetic field on M_0. If we define $\mathcal{H}(q,p) = \mathcal{H}_0(q, \delta_q(p, A(q)))$, then $\mathcal{H}$ is the Hamiltonian of a charged particle in M_0. Thus A defines a new zero of energy-momentum and the charged particle Hamiltonian is simply the free Hamiltonian modified by choosing a new zero of energy-momentum at each point. The old physical vacuum is redefined to obtain a new vacuum via an electromagnetic field. These ideas are discussed fully in [9] where physical arguments are given to support the use of affine structures such as these in charged particle dynamics.

In addition to this example relating to charged particle dynamics there are tantalizing hints that affine geometry is implicitly utilized elsewhere in physics. Norris discussed Newtonian Mechanics within such a context in [13]. A more trivial but intriguing example occurs in relativistic electromagnetism. At each point q of Minkowski space M_0 let C_q denote the past null light cone at q. Let $C = \bigcup_{q \in M_0} C_q$ and define δ on C by $\delta_q(x,y) = \vec{x} - \vec{y}$ where $x = (x^0, \vec{x})$, $y = (y^0, \vec{y})$ are elements of $C_q \subseteq M_0$. Then δ is a difference function field on C modelled on the trivial vector bundle with standard fiber $\mathbf{R}^3$. Define a section θ of C by $\theta(C_q) = q$. Given a current J on M_0 the retarded potential associated with J is defined at $x \in M_0$ by

$$A_i^{\mathrm{ret}}(x) = \int_{\mathbf{R}^3} J_i(x^0 - |\vec{y}|,\ \vec{x} + \vec{y}) \left(\frac{d^3\vec{y}}{|\vec{y}|} \right).$$

Here $d^3\vec{y}$ denotes Lebesgue measure on $\mathbf{R}^3$ and $\frac{d^3\vec{y}}{|\vec{y}|}$ is the Lorentz invariant measure on the cone C_x induced by the mapping $\vec{y} \to (x^0 - |\varphi_x(y)|,\ \varphi_x(y))$ where $\varphi_x : C_x \to \mathbf{R}^3$ is defined by $\varphi_x(y) = \vec{y} + \vec{x}$. Note that the mapping φ_x has the property that it defines the difference function field δ_x, i.e., $\delta_x(y,z) = \varphi_x(y) - \varphi_x(z)$ for

$y, z \in C_x$. Observe also that $x \to \varphi_x$ is a family of fiber diffeomorphisms related to the affine structure δ of the affine bundle C as in Section 5. It would be interesting to reformulate the theory of retarded and advanced potentials in terms of affine geometrical concepts and to investigate whether such a formulation yields new physical insight into this somewhat murky area. Such an investigation is beyond the scope of this paper and probably needs the attention of a physicist.

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